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Abstract

This paper proposes a dynamic equilibrium model that can provide a unified explanation for the stylized facts observed in stock index markets such as the fat tails of risk-neutral return distribution relative to physical distribution, negative expected returns on deep OTM call options, and negative realized variance risk premiums. In particular, we focus on the U-shaped pricing kernel against the stock index return, which is closely related to the negative call returns. We assume that the stock index return follows the time-changed Lévy process and that a representative investor has power utility over the aggregate consumption that forms a linear regression of the stock index return and its stochastic activity rate. This model offers a macroeconomic interpretation of the stylized facts from the perspective of the sensitivity of the activity rate and stock index return on the aggregate consumption as well as the investor's risk aversion.

Keywords: stock index, U-shaped pricing kernel, stochastic activity rate, aggregate consumption, physical distribution, risk-neutral distribution, realized variance

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1 Introduction

Pricing kernels are a central issue in the study of asset pricing theory. Since a pricing kernel is defined as the ratio of risk-neutral probability to physical probability, it is crucial to know the detailed features and relations of physical and risk-neutral distributions. So far, several stylized facts associated with pricing kernels have been observed in stock index markets such as S&P 500 and NIKKEI 225 among others. A number of empirical and theoretical studies have already addressed such stylized facts. Some of them[1] are as follows.

The tail of the risk-neutral density of a stock index return is fatter than that of the physical density and the skewness of the risk-neutral return distribution is more negative than that of the physical distribution. See for instance Rubinstein (1984), Jackwerth & Rubinstein (1996), Ait-Sahalia & Lo (2000), Jackwerth (2000), Carr et al. (2002), Rosenberg & Engle (2002), Bliss & Panigirtzoglou (2004), and Rompolis & Tzavalis (2008). In particular, our study is closely related to Bakshi et al. (2002), which showed how risk aversion induces negative skewness in the risk-neutral distribution of the market return.

As regards theoretical research, the pricing kernels monotonically decreasing in aggregate wealth were first developed by Lucas (1978). Thereafter, the standard asset pricing theory has basically relied on such pricing kernels. However, some empirical analyses such as Ait-Sahalia & Lo (2000), Jackwerth (2000), and Rosenberg & Engle (2002) have demonstrated that pricing kernels plotted against the market return are not monotonically decreasing, but have some increasing region. That is, they discovered U-shaped pricing kernels. More recently, several sophisticated studies as well as theoretical and empirical approaches have considered the U-shaped pricing kernels. For instance, Bakshi et al. (2010) investigated the theoretical relation between the expected returns of contingent claims with payout on the upside and the slope of pricing kernels against the index return. They asserted that U-shaped pricing kernels can explain the negative expected returns of deep OTM calls, which is an anomaly found in major index option markets. In addition, they proposed a model in which the agents have heterogeneity in belief about the market return and illustrated that this duplicates the U-shaped pricing kernel. However, we would like to stress that they consider only a single-period static model. Applying the GARCH option pricing model developed by Heston & Nandi (2000), Christoffersen et al. (2013) directly introduced the pricing kernel, which is monotonic in the market return and in its variance, although the projection of the pricing kernel onto the market return is U-shaped. Then, they demonstrated that the pricing kernel can reconcile the time series data of the market return with the observed option prices. Moreover, their model could reproduce the overreaction of long-term options to changes in short-term realized volatility and the fat tails of market return distribution under the risk-neutral measure. Using a nonparametric approach, Song & Xiu (2016) estimated the marginal pricing kernels conditional on VIX and the term structure of variance swaps on S&P 500. They concluded that the pricing kernel of the market return exhibits an upward-slope in the extreme end of the right tail. Furthermore, they showed that the pricing kernel plotted against VIX is considerably U-shaped.

A number of studies considered the index option returns and their anomalies; for example, Merton et al. (1978), Rubinstein (1984), Coval & Shumway (2001), Pan (2002), Vanden (2004), Cao & Huang (2008), and Broadie et al. (2009). In particular, our preoccupation is with the stylized fact that the average returns of deep OTM call options on the stock index are usually negative. As mentioned above, this anomaly is closely related to the U-shaped pricing kernels. We are also concerned with the stylized fact that the implied volatilities observed in

[1] See Pennacchi (2008), Cochrane (2009), Duffie (2010), and Back (2010) as excellent monographs for the standard theory of asset pricing.

index option markets tend to exceed the realized volatilities. This fact can be explained by the negative price of variance risk known as the negative variance risk premium. See for example Carr and Wu (2009), Bollerslev et al. (2009), Todorov (2010), and Bollerslev et al. (2014), which examined the negative variance risk premium in detail.

The purpose of our study is to propose a dynamic equilibrium model that can provide a unified explanation of the stylized facts such as distortion of risk-neutral distribution of the index return to physical distribution, negative expected returns of deep OTM call options on the stock index, and negative risk premiums on the realized variance of the index return. Particularly, our important concern is the endogenous replication of the U-shaped pricing kernels against the index return. Then, we offer a macroeconomic interpretation of the stylized facts in the following steps.

First, we set up an equilibrium model in continuous time; this can be regarded as a consumption-based asset pricing model. See for instance Eraker (2008), Martin (2013), and Yamazaki (2015, 2017), recent work taking the consumption-based asset pricing model. In the model, the representative investor has power utility over aggregate consumption and the logarithm of aggregate consumption is represented as a linear combination of the stock index return and its variance. The rate of return on stock index is assumed to follow the time-changed Lévy process proposed by Carr et al. (2003) and Carr and Wu (2004). Time-changed Lévy processes are an extension of Lévy processes with stochastic time change. They provide a flexible framework for generating jumps, capturing stochastic volatility as the random time change, and introducing negative correlation between stock returns and their volatility. In addition, they produce a vast class of stochastic processes including Brownian motions, pure jump processes, and traditional stochastic volatility models, as well as a number of peculiar stochastic processes such as the VG-CIR model and the NIG Γ-OU model[2].

Second, we drive formulas to evaluate the expected return of the stock index, the term structure of interest rates, call option prices on the stock index, and variance swap rates of the stock index in market equilibrium from the model. Furthermore, we obtain analytic representations of the fundamental statistics of both physical and risk-neutral distributions of the stock index return and the projection of pricing kernels onto the stock index return. As a general result, all the formulas are represented as consisting of the characteristic function of the time-changed Lévy process.

Third, we examine some concrete models to confirm the possibility of a unified explanation of the stylized facts with which we are concerned. In the case of Lévy processes with constant time change such as the Black-Scholes model, the variance gamma model, and the normal inverse Gaussian model, we can analytically investigate causality of the stylized facts. As is well known, the Black-Scholes model only changes the expected rate of return on the stock index to the interest rate by change of measure. In contrast, we show that the variance gamma model and normal inverse Gaussian model can induce negative skewness and excess kurtosis of the risk-neutral return distributions by change of measure due to the investor's risk aversion. Regrettably, these models cannot depict any U-shaped pricing kernels. That is, the pricing kernels in these models are monotonically decreasing in the index returns. On the other hand, in the case of proper time-changed Lévy processes such as Heston's stochastic volatility model, dynamic equilibrium models can replicate the stylized facts. Numerical examples show that risk-neutral return distributions have more negative skewness and excess kurtosis than physical distributions. The implied volatilities are strongly skewed even when the stock index is not correlated with its variance. We demonstrate that if the aggregate consumption is considerably decreasing in variance of the index return, then the projection of the pricing kernel onto the index returns is U-shaped. The negative expected returns of deep OTM call options on the

[2]See Schoutens (2003) for example.

stock index result from the U-shaped pricing kernels. This affirms the validity of the result in Bakshi et al. (2010) even though the model is dynamic in continuous time.

The remainder of the paper is organized as follows. Section 2 discusses the setup of the dynamic equilibrium model. Section 3 derives general results replicating the stylized facts observed in stock index markets from the model. Section 4 demonstrates the case of the stock index modeled by classical geometric Lévy processes, and Section 5 addresses the case of time-changed Lévy processes with stochastic time change. Section 6 provides numerical examples, and our concluding remarks are presented in Section 7. The appendix gives almost all proofs of propositions.

2 Setup

2.1 Time-Changed Lévy Process

A càdlàg stochastic process $(Y_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with values in $\mathbf{R}$ such that $Y_0 = 0$ is called a *Lévy process* if it possesses the following properties: (1) For every $t_1 < \cdots < t_n$, the increments $Y_{t_2} - Y_{t_1}, \ldots, Y_{t_n} - Y_{t_{n-1}}$ are independent. (2) For any $h > 0$, $Y_{t+h} - Y_t$ has the same law as X_h. (3) For any $\varepsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|Y_{t+h} - X_t| \geq \varepsilon) = 0$.

Lemma 1 *(Lévy-Khintchine formula) Let $(Y_t)_{t\geq 0}$ be a Lévy process on $\mathbf{R}$ under a measure $\mathbb{P}$. The characteristic function of the distribution of Y_t has the form*

$$\Phi^{\mathbb{P}}_{Y_t}(\theta) := \mathbb{E}^{\mathbb{P}}\left[e^{i\theta Y_t}\right] = e^{t\varphi_Y(\theta)},$$

where $\theta \in \mathbf{R}$, $\mathbb{E}^{\mathbb{P}}[\,\cdot\,]$ denotes the expectation operator under $\mathbb{P}$, and the function φ_Y called the characteristic exponent is given by

$$\varphi_Y(\theta) = i\alpha\theta - \frac{1}{2}\beta\theta^2 + \int_{-\infty}^{\infty} \left(e^{i\theta y} - 1 - i\theta y \mathbf{1}_{\{|y|\leq 1\}}\right) \nu_Y(dy).$$

where $\alpha \in \mathbf{R}$ and $\beta \geq 0$ are constants, and ν_Y is a positive Radon measure on $\mathbf{R}\setminus\{0\}$ satisfying

$$\int_{-\infty}^{\infty} (1 \wedge y^2)\nu_Y(dy) < \infty.$$

Parameter β is called the *Gaussian coefficient* and the measure ν_Y is known as the *Lévy measure.* The Gaussian coefficient β is the constant variance of the continuous component of the Lévy process and the Lévy measure ν_Y determines its jump structure.

A *time-changed Lévy process* is a stochastic process $(X_t)_{t\geq 0}$ defined as

$$X_t = Y_{\tau_t}, \qquad t \geq 0, \tag{2.1}$$

where $(Y_t)_{t\geq 0}$ is a Lévy process and $(\tau_t)_{t\geq 0}$ is an increasing càdlàg process adapted to $(\mathcal{F}_t)_{t\geq 0}$ such that $\tau_t \to \infty$ as $t \to \infty$. Moreover, we assume that the random time τ_t is given by

$$\tau_t = \int_0^t v_{s-} ds, \qquad \text{for every } \ t \geq 0,$$

where $(v_t)_{t\geq 0}$ called the *activity rate* is a non-negative càdlàg process with values in $\mathbf{R}$. Without loss of generality, we put $v_0 = 1$ throughout this paper.

Carr et al. (2003) and Carr & Wu (2004) originally proposed the time-changed Lévy processes for modeling dynamics of an asset price in order to introduce the concept of stochastic volatilities into Lévy processes. Intuitively, τ_t can be regarded as business time at calendar time t. A more active business day, on which the corresponding active rate becomes higher, generates higher volatility in a market. This randomness in business activity induces randomness in volatility. It is worthwhile noting that the time-changed Lévy processes can produce various types of discontinuous processes with stochastic volatility by combining an arbitrary Lévy process $(Y_t)_{t\geq 0}$ with an arbitrary activity rate $(v_t)_{t\geq 0}$.

2.2 Model

We assume a frictionless stock index market including its options and variance swaps, in which there are no arbitrage opportunities. We denote $\mathbb{P}$ as the physical probability measure in the market. There is a representative investor in the market, who has a power utility function over an aggregate consumption stream $(C_t)_{t\geq 0}$. Thus, the investor's expected utility is given by

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^{\infty} e^{-\delta t}\frac{C_t^{1-\gamma}}{1-\gamma}dt\right], \tag{2.2}$$

where the rate of time preference δ and the relative risk aversion γ are non-negative constants. Let S_t be the level of the stock index at time t. Suppose that the log-return on the stock index at time t is given by

$$R_t = \log\frac{S_t}{S_0} := \mu(t) + X_t - \varphi_Y(-i)\tau_t,$$

where $\mu(t)$ is a deterministic function of time t. As will be seen, the process $(e^{X_t-\varphi_Y(-i)\tau_t})_{t\geq 0}$ is a $\mathbb{P}$-martingale. Consequently, $\mu(t)$ must be the expected cumulative rate of return on the stock index and will be endogenously determined in market equilibrium. Next, suppose that the aggregate consumption stream $(C_t)_{t\geq 0}$ is given by[3]

$$d\log C_t = \hat{a}dR_t + \hat{b}dv_t,$$

where the coefficients $\hat{a}$ and $\hat{b}$ are constants. Thus, the log-consumption equivalent to the growth rate of the aggregate consumption is formulated as a linear regression model with the two explanatory variables R_t and v_t. There is a natural hypothesis that the coefficient $\hat{a}$ is positive because the consumption level is usually high when the level of the stock index is high, and vice versa. It is plausible that the coefficient $\hat{b}$ takes a negative value because increasing uncertainty in the stock market might deteriorate the aggregate consumption.

3 General Results

Carr & Wu (2004) proved that the characteristic function of the distribution of X_t defined in (2.1) cab be represented as

$$\Phi^{\mathbb{P}}_{X_t}(\theta) = \mathbb{E}^{\mathbb{P}}\left[e^{i\theta Y_{\tau_t}}\right] = \mathbb{E}^{\mathbb{P}(\theta)}\left[e^{\tau_t\varphi_Y(\theta)}\right],$$

[3] As more realistic modeling, we can formulate

$$d\log C_t = \hat{a}dR_t + \hat{b}dv_t + d\epsilon_t,$$

where $(\epsilon_t)_{t\geq 0}$ is an independent noise process such that $\mathbb{E}^{\mathbb{P}}[\epsilon_t] = 0$ for all $t \geq 0$. However, in this paper we omit such a noise process for simplicity. Even if the noise process is introduced to the model, we can easily develop the formulas as a simple extension of the result in this paper.

where $\mathbb{E}^{\mathbb{P}(\theta)}[\,\cdot\,]$ denotes the expectation operator under new measure $\mathbb{P}(\theta)$, which is absolutely continuous with respect to the physical measure $\mathbb{P}$ and defined by a complex-valued exponential martingale

$$Z_T(\theta) := \left.\frac{d\mathbb{P}(\theta)}{d\mathbb{P}}\right|_T = \exp\left\{i\theta X_T - \varphi_Y(\theta)\tau_T\right\}.$$

Thus, $Z_T(\theta)$ is the Radon-Nikodym derivative of the complex-valued measure $\mathbb{P}(\theta)$ with respect to the physical measure $\mathbb{P}$ up to time T. The optional stopping theorem ensures that

$$Z_t(\theta) = \mathbb{E}_t^{\mathbb{P}}\left[Z_T(\theta)\right] = \exp\left\{i\theta X_t - \varphi_Y(\theta)\tau_t\right\},$$

is a $\mathbb{P}$-martingale, where $\mathbb{E}_t^{\mathbb{P}}[\,\cdot\,]$ denotes the expectation operator conditional on $\mathcal{F}_t$ under $\mathbb{P}$. Furthermore, for an arbitrary $\mathcal{F}_T$-measurable random variable U on $(\Omega, \mathcal{F}, \mathbb{P})$, it satisfies

$$\mathbb{E}_t^{\mathbb{P}(\theta)}\left[U\right] = \mathbb{E}_t^{\mathbb{P}}\left[\frac{Z_T(\theta)}{Z_t(\theta)}U\right], \tag{3.1}$$

for all $t \in [0, T]$. If the Lévy process $(Y_t)_{t\geq 0}$ is independent of the random time $(\tau_t)_{t\geq 0}$, such a change of measure is unnecessary and the characteristic function of X_t is given by

$$\Phi_{X_t}^{\mathbb{P}}(\theta) = \mathbb{E}^{\mathbb{P}}\left[e^{\tau_t\varphi_Y(\theta)}\right].$$

Note that if $\theta = 0$ the new measure $\mathbb{P}(\theta)$ is coincident with the original physical measure $\mathbb{P}$.

The following simple lemma will be frequently used for parsimonious notations and plays an important role throughout this paper.

Lemma 2 *Suppose that the bivariate characteristic function of the joint distribution of* $(X_t - \varphi_Y(-i)\tau_t, v_t)$

$$\Psi_t(\theta_1, \theta_2) := \mathbb{E}^{\mathbb{P}}\left[\exp\left\{i\theta_1\left(X_t - \varphi_Y(-i)\tau_t\right) + i\theta_2 v_t\right\}\right],$$

is well-defined for both complex and real arguments on some region of $\mathbf{C}^2$*. Then, we have*

$$\Psi_t(\theta_1, \theta_2) = \mathbb{E}^{\mathbb{P}(\theta_1)}\left[\exp\left\{\lambda(\theta_1)\tau_t + i\theta_2 v_t\right\}\right], \tag{3.2}$$

where we define

$$\lambda(x) := \varphi_Y(x) - ix\varphi_Y(-i). \tag{3.3}$$

Proof of Lemma 2: Using the relation (3.1), we obtain (3.2). □

Denote the cumulant generating function of the joint distribution of $(X_t - \varphi_Y(-i)\tau_t, v_t)$ by

$$\psi_t(\theta_1, \theta_2) := \log \Psi_t(-i\theta_1, -i\theta_2). \tag{3.4}$$

As will be shown, the general results we derive below are represented by equations composed of the characteristic function Ψ_t or the cumulant generating function ψ_t. Note that the right side of (3.2) is regarded as a characteristic function of the bivariate distribution of $(\int_0^t v_s ds, v_t)^\top$ under $\mathbb{P}(\theta_1)$. Section 5 will provide the closed-form expressions of (3.2) in some specific models.

3.1 Expected Return on Stock Index and Interest Rate

As the representative investor has the power utility over the aggregate consumption given in (2.2), the stochastic discount factor in the market is given by the form

$$M_t := e^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-\gamma} = \exp\left\{\hat{b}\gamma - \delta t - \hat{a}\gamma\mu(t) - \hat{a}\gamma X_t + \hat{a}\gamma\varphi_Y(-i)\tau_t - \hat{b}\gamma v_t\right\}. \tag{3.5}$$

Note that the stochastic discount factor M_t depends on the log-return on the stock index R_t, which is itself endogenous to the model.

Christoffersen et al. (2013) directly formulated a pricing kernel under Heston's stochastic volatility model (Heston, 1993) that is similar to (3.5). In Christoffersen et al. (2013), $-\hat{a}\gamma$ and $-\hat{b}\gamma$ were treated as one parameters and called *risk-aversion parameter* and *variance preference parameter*, respectively. Moreover, they showed by substantial empirical analysis that variance preference parameter $-\hat{b}\gamma$ is positive, which is consistent with the hypothesis we mentioned above that aggregate consumption is decreasing when uncertainty in the stock market is high. From now on, we summarize these parameters as $a := \hat{a}\gamma$ and $b := \hat{b}\gamma$ along the lines of Christoffersen et al. (2013) and assume $a \neq 1$ for a technical constraint.

Our approach is, however, different from Christoffersen et al. (2013). We are concerned with the time-changed Lévy processes that can be regarded as a sufficiently large class for modeling asset price dynamics including Heston's stochastic volatility model. Our approach is based on the characteristic functions and the cumulant generating functions for investigation of stylized facts observed in stock index markets. We will endogenously determine the expected rate of the stock index return,the term structure of interest rates, and the equity risk premium in market equilibrium, while Christoffersen et al. (2013) exogenously gave them. In addition, we will examine not only the index options, but also the variance swaps on the stock index.

First of all, we derive the equilibrium expected return on the stock index.

Proposition 1 *(Expected Return on Stock Index) The cumulative expected rate of return on the stock index is given by*

$$\mu(t) = \frac{\delta t - b}{1-a} - \frac{1}{1-a}\psi_t\left(1-a, -b\right). \tag{3.6}$$

Proof of Proposition 1: Following to the Euler equation, the level of the stock index should satisfy $S_0 = \mathbb{E}^{\mathbb{P}}\left[M_t S_t\right]$ for all $t \geq 0$. Namely, we have

$$\mathbb{E}^{\mathbb{P}}\left[M_t\left(\frac{S_t}{S_0}\right)\right] = \exp\left\{b - \delta t - (a-1)\mu(t)\right\}\Psi_t\left(i(a-1), ib\right) = 1.$$

Therefore, the relation

$$b - \delta t - (a-1)\mu(t) + \psi_t\left(1-a, -b\right) = 0,$$

holds for every $t \geq 0$. □

Let $B(t)$ be the price of a zero-coupon risk-free bond at time zero that pays one unit at maturity t. Define the interest rate as $r(t) := -\log B(t)$, which probably should be called the cumulative instantaneous interest rate more accurately. The next proposition presents the equilibrium interest rate.

Proposition 2 *(Interest Rate) The interest rate is given by*

$$r(t) = \frac{\delta t - b}{1-a} - \frac{a}{1-a}\psi_t(1-a, -b) - \psi_t\left(-a, -b\right). \tag{3.7}$$

Proof of Proposition 2: The Euler equation implies that the price of a zero-coupon bond at time zero is given by

$$B(t) = \mathbb{E}^{\mathbb{P}}\left[M_t\right] = \exp\left\{b - \delta t - a\mu(t)\right\}\Psi_t\left(ia, ib\right).$$

Plugging (3.6) into the above equation, we obtain (3.7). □

Corollary 1 *(Equity Risk Premium) The equity risk premium on the stock index is given by*

$$\mu(t) - r(t) = \psi_t\left(-a, -b\right) - \psi_t\left(1-a, -b\right). \tag{3.8}$$

Here, we briefly mention some remarks below. As $a \to 1$, the absolute value of the expected return $\mu(t)$ diverges to infinity. This means that no arbitrage conditions are violated when $a = 1$. Next, note that $\psi_t(0,0) = 0$ due to the definition and $\psi_t(1,0) = 0$ because $(e^{X_t - \varphi_Y(-i)\tau_t})_{t\geq 0}$ is a $\mathbb{P}$-martingale. Therefore, we confirm that when the representative investor is risk-neutral, that is, $\gamma = 0$, the equity risk premium becomes zero and $r(t) = \mu(t) = \delta t$. This is of course consistent.

3.2 Physical and Risk-Neutral Distribution of Stock Index Return

In this subsection, we derive some useful formulas to examine properties of the physical and risk-neutral distributions of the log-return on the stock index. Our concern is stylized facts observed between time series of a stock index and the shape of implied volatilities observed in its option market. Moreover, there is a controversial problem that the shape of the projection of the pricing kernel onto the return of the stock index: Whether is the pricing kernel monotonically decreasing or U-shaped in the stock index return?

The characteristic function of the physical distribution of the log-return on the stock index is easily obtained as

$$\Phi^{\mathbb{P}}_{R_t}(\theta) := \mathbb{E}^{\mathbb{P}}\left[e^{i\theta R_t}\right] = e^{i\theta\mu(t)}\Psi_t\left(\theta, 0\right). \tag{3.9}$$

The risk-neutral probability in the market is defined by the Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_T := \frac{M_T}{\mathbb{E}^{\mathbb{P}}\left[M_T\right]}, \tag{3.10}$$

which is just the pricing kernel we would to discuss precisely below. The following proposition provides the characteristic function of the risk-neutral distribution of R_t.

Proposition 3 *(Risk-Neutral Characteristic Function) The characteristic function of the risk-neutral distribution of the log-return on the stock index defined as*

$$\Phi^{\mathbb{Q}}_{R_t}(\theta) := \mathbb{E}^{\mathbb{Q}}\left[e^{i\theta R_t}\right],$$

is given by

$$\Phi^{\mathbb{Q}}_{R_t}(\theta) = \frac{e^{i\theta\mu(t)}}{\Psi^0_t}\Psi_t\left(\theta + ia, ib\right), \tag{3.11}$$

where we define the time-t normalization factor as $\Psi^0_t := \Psi_t\left(ia, ib\right)$.

Proof of Proposition 3: It can be easily checked that

$$\begin{aligned}\Phi^{\mathbb{Q}}_{R_t}(\theta) &= \mathbb{E}^{\mathbb{Q}}\left[e^{i\theta R_t}\right] = e^{r(t)}\mathbb{E}^{\mathbb{P}}\left[M_t e^{i\theta R_t}\right] \\ &= \exp\{r(t) - \delta t + (i\theta - a)\mu(t) + b\}\,\Psi_t(\theta + ia, ib)\,. \end{aligned} \tag{3.12}$$

From Corollary 1 and Proposition 1, we have

$$r(t) - \delta t + (i\theta - a)\mu(t) + b = i\theta\mu(t) - \psi_t(-a, -b). \tag{3.13}$$

Substituting (3.13) into (3.12), the characteristic function (3.11) is obtained. □

In the standard theory of asset pricing in continuous time such as Duffie et al. (2000), Eraker & Shaliastovich (2008), and Christoffersen et al. (2013), dynamics of underlying asset returns under physical and risk-neutral measure are characterized by the corresponding each system of stochastic differential equations. In contrast, we try to characterize the dynamics of the stock index return by the physical and risk-neutral characteristic functions.

To compare the risk-neutral probability distribution with the physical one, their cumulant generating functions are useful. The cumulant generating functions of the physical and the risk-neutral distribution are given by

$$\phi^{\mathbb{P}}_{R_t}(\theta) := \log\Phi^{\mathbb{P}}_{R_t}(-i\theta) = \theta\mu(t) + \psi_t(\theta, 0),$$

and

$$\phi^{\mathbb{Q}}_{R_t}(\theta) := \log\Phi^{\mathbb{Q}}_{R_t}(-i\theta) = \theta\mu(t) + \psi_t(\theta - a, -b) - \psi_t(-a, -b),$$

respectively. Consequently, the nth cumulants of each distribution

$$c^{\mathbb{P}}_n(R_t) := \frac{d^n}{d\theta^n}\phi^{\mathbb{P}}_{R_t}(\theta)\bigg|_{\theta=0} \qquad \text{and} \qquad c^{\mathbb{Q}}_n(R_t) := \frac{d^n}{d\theta^n}\phi^{\mathbb{Q}}_{R_t}(\theta)\bigg|_{\theta=0}$$

can be computed. Empirical analysis such as Carr et al. (2000), Bakshi et al. (2003), and Broadie et al. (2007) documented the stylized fact that the risk-neutral standard deviation of stock index returns exceeds the physical one, that is,

$$c^{\mathbb{P}}_2(R_t) < c^{\mathbb{Q}}_2(R_t). \tag{3.14}$$

This means that the variance of the stock index returns has a negative risk premium. They also pointed out that the risk-neutral skewness is negative, whereas the physical skewness is nearly equal to zero.

$$c^{\mathbb{P}}_3(R_t) \approx 0 \quad \text{and} \quad c^{\mathbb{Q}}_3(R_t) < 0. \tag{3.15}$$

Furthermore, they reported that the risk-neutral kurtosis exceeds or nearly equals the physical kurtosis. With the stylized fact (3.14), this is necessary to be

$$c^{\mathbb{P}}_4(R_t) < c^{\mathbb{Q}}_4(R_t). \tag{3.16}$$

In the next section, we seek suitable specific models belonging to the class of time-changed Lévy process and persuasive parameter sets to be able to reconcile the stylized facts we mention above.

Our preoccupation is the shape of the projection of the pricing kernel (3.10) onto the log-return on the stock index. It can be calculated by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(R_t < x) := \frac{\mathfrak{F}^{-1}\left[\Phi^{\mathbb{Q}}_{R_t}\right](x)}{\mathfrak{F}^{-1}\left[\Phi^{\mathbb{P}}_{R_t}\right](x)}, \tag{3.17}$$

where $\mathfrak{F}^{-1}[f](x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x} f(\theta)dx$ is the inverse Fourier transform of a function f. The projection of the pricing kernel (3.17) is defined under the assumption that there exist the density functions of both the physical and risk-neutral distributions. Recent studies such as Bakshi et al. (2010) and Christoffersen et al. (2013) demonstrated that the projection of the pricing kernel (3.17) depicts U-shape curve, while the classical asset pricing theory concludes that the pricing kernel is monotonically decreasing function with respect to asset returns.

3.3 Stock Index Option

The price of a call option on the stock index with maturity T and strike K is given by

$$C(T,K) := \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T)}(S_T - K)^+\right] = S_0 e^{-r(T)}\mathbb{E}^{\mathbb{Q}}\left[\left(e^{R_T} - e^k\right)^+\right], \tag{3.18}$$

and the expected payoff of the call option is given by

$$\mathbb{E}^{\mathbb{P}}\left[(S_T - K)^+\right] = S_0\mathbb{E}^{\mathbb{P}}\left[\left(e^{R_T} - e^k\right)^+\right], \tag{3.19}$$

where $x^+ = \max(x, 0)$ and $k := \log(K/S_0)$. The expected log-return of the option is defined as

$$\log\left(\frac{\mathbb{E}^{\mathbb{P}}\left[(S_T - K)^+\right]}{C(T,K)}\right) = \log\mathbb{E}^{\mathbb{P}}\left[\left(e^{R_T} - e^k\right)^+\right] - \log\mathbb{E}^{\mathbb{Q}}\left[\left(e^{R_T} - e^k\right)^+\right] + r(T). \tag{3.20}$$

About a put option, its price is obtained by the put-call parity and its expected payoff is given by the following relation,

$$\mathbb{E}^{\mathbb{P}}\left[(K - S_T)^+\right] = \mathbb{E}^{\mathbb{P}}\left[(S_T - K)^+\right] - S_0 e^{\mu(T)} + K.$$

The expected log-return of the put option is defined in a similar way to (3.20). Therefore, for our option analysis we only need to have the expectations on the right sides of the equations (3.18) and (3.19), which are able to be computed by the following proposition.

Proposition 4 *(Expected Payoff of Call) The expected payoffs of a call option on the stock index with maturity T and strike K under the physical and risk-neutral measure are given by*

$$\mathbb{E}^{\mathbb{P}}\left[\left(e^{R_T} - e^k\right)^+\right] = \mathfrak{F}^{-1}[g_T^{\mathbb{P}}](k) + \left(e^{\mu(T)} - e^k\right)^+, \tag{3.21}$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\left(e^{R_T} - e^k\right)^+\right] = \mathfrak{F}^{-1}[g_T^{\mathbb{Q}}](k) + \left(e^{r(T)} - e^k\right)^+, \tag{3.22}$$

respectively. Here, we define

$$g_T^{\mathbb{P}}(\theta) = \frac{\Phi_{R_T}^{\mathbb{P}}(\theta - i) - e^{(i\theta+1)\mu(T)}}{i\theta(i\theta+1)}, \quad \textit{and} \quad g_T^{\mathbb{Q}}(\theta) = \frac{\Phi_{R_T}^{\mathbb{Q}}(\theta - i) - e^{(i\theta+1)r(T)}}{i\theta(i\theta+1)}.$$

The proof of Proposition 4 can be found in Appendix A.1.

Bakshi et al. (2010) have proved the following important theoretical result for the stylized facts about the relationship between expected returns of call options on the stock index level and the shape of the projection of the pricing kernel onto the stock index return.

Theorem 1 *(Bakshi et al., 2010) If the economy supports a U-shaped pricing kernel, then the following statements are true:*

1. *There exists a strike price K_d such that expected returns of call options on the stock index level with strikes $K > K_d$ are decreasing in K.*
2. *There exists a strike price K_n such that call options with strikes higher than K_n have negative expected returns.*
3. *The steeper the slope of the U-shaped pricing kernel in the increasing region, the more negative are the expected returns of call options with strikes higher than K_n.*

The above theorem is very general without any specifications of dynamics of the stock index level. Our purpose is to examine a suitable stochastic process as a driving factor of the stock index and to provide a concrete dynamic asset pricing model for reverse-engineering the situation in Theorem 1

3.4 Realized Variance of Stock Index Return

The realized variance of the stock index denoted by V_T is defined as the sum of squared log-returns of the stock index

$$V_T := \sum_{l=1}^{L} \left(\log \frac{S_{t_l}}{S_{t_{l-1}}} \right)^2 = \sum_{l=1}^{L} (\Delta R_l)^2 \,, \tag{3.23}$$

where $0 = t_0 < t_2 < \cdots < t_L = T$ and $\Delta R_l := R_{t_l} - R_{t_{l-1}}$. For simplicity, we assume equal intervals of the monitoring dates, that is, $t_l := l\Delta t$ for $l = 0, 1, \ldots, L$ where $\Delta t = T/L$.

The variance swap rate is the expected realized variance under the risk-neutral measure $\mathbb{E}^{\mathbb{Q}}[V_T]$. In general, the variance swap rate takes a different value form the expected realized variance under the physical measure $\mathbb{E}^{\mathbb{P}}[V_T]$. Actually, the stylized fact that the variance swap rate exceeds the expected realized variance under the physical measure has been observed in variance swap markets. That is,

$$\mathbb{E}^{\mathbb{P}}[V_T] < \mathbb{E}^{\mathbb{Q}}[V_T]. \tag{3.24}$$

Note that the stylized fact (3.24) is not necessarily equivalent to the stylized fact (3.14). For understanding these stylized facts more deeply, it is important to perceive the differences between the expected realized variance and the variance of the distribution of the log-return under each measure. They can be represented as

$$\mathbb{E}^{\mathbb{P}}[V_T] - \mathbf{Var}^{\mathbb{P}}[R_T] = \mathbb{E}^{\mathbb{P}}[R_T]^2 - 2 \sum_{m<n} \mathbb{E}^{\mathbb{P}}[\Delta R_m \Delta R_n], \tag{3.25}$$

and

$$\mathbb{E}^{\mathbb{Q}}[V_T] - \mathbf{Var}^{\mathbb{Q}}[R_T] = \mathbb{E}^{\mathbb{Q}}[R_T]^2 - 2 \sum_{m<n} \mathbb{E}^{\mathbb{Q}}[\Delta R_m \Delta R_n], \tag{3.26}$$

respectively. Here, $\mathbf{Var}^{\mathbb{P}}[\,\cdot\,]$ and $\mathbf{Var}^{\mathbb{Q}}[\,\cdot\,]$ denote the variance operators under the physical and the risk-neutral measures, respectively. Namely, the differences (3.25) and (3.26) accrue from the autocorrelation of increments of log-returns under the corresponding measures. The next proposition analytically representing the characteristic functions of the distributions of an increment of log-returns is useful to compute the expected realized variance.

Proposition 5 *(Characteristic Function of an Increment of Log-Returns) The characteristic function of the distributions of each increment of the stock index log-returns under the physical and risk-neutral measure are given by*

$$\Phi^{\mathbb{P}}_{\Delta R_l}(\theta) = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}(\theta)}_{t_{l-1}}\left[\exp\left\{\lambda(\theta)\int_{t_{l-1}}^{t_l} v_{s-}ds\right\}\right]\right], \tag{3.27}$$

and

$$\begin{aligned}\Phi^{\mathbb{Q}}_{\Delta R_l}(\theta) &= \frac{e^{i\theta(\mu(t_l)-\mu(t_{l-1}))}}{\Psi^0_{t_l}} \times \\ &\mathbb{E}^{\mathbb{P}(ia)}\left[e^{\lambda(ia)\tau_{t_{l-1}}}\mathbb{E}^{\mathbb{P}(\theta+ia)}_{t_{l-1}}\left[\exp\left\{\lambda(\theta+ia)\int_{t_{l-1}}^{t_l} v_{s-}ds - bv_{t_l}\right\}\right]\right],\end{aligned} \tag{3.28}$$

respectively.

The proof of Proposition 5 is put in Appendix A.2. If the characteristic function of the joint distribution of $(\tau_t, v_t)^\top$ is represented as an exponential affine form, the closed-form expressions of (3.27) and (3.28) can be obtained. The next section presents some examples such that (3.27) and (3.28) have closed-form expressions. In order to compute the expected realized variance $\mathbb{E}^{\mathbb{P}}[V_T]$ and $\mathbb{E}^{\mathbb{Q}}[V_T]$, we only need to know the second moment of an arbitrary increment of the stock index log-returns, that is, the twice differentials of the characteristic functions (3.27) and (3.28) at $\theta = 0$.

Properties of the physical and the risk-neutral distributions of the realized variance are able to be examined in terms of their moments or cumulants. Appendix B provides a general analytic method for computing any moment of the realized variance. However, we omit to examine such properties for specific models because the relationship between the physical and the risk-neutral distributions of the realized variance other than (3.24) have not been obvious. In addition to the above, tedious calculations are needed to obtain the realized variance moments from the method developed in Appendix B.

4 Constant Activity Rate

In this section, we consider the case that the log-return on the stock index is governed by a Lévy process, that is, $v_t = 1$, $\tau_t = t$, and $X_t = Y_t$ for all $t \geq 0$. This model can be regarded as a special case that the underlying process does not equipped with stochastic time change. As a result, the change of measure defined by (3.10) is reduced to the Esscher transform. Thus, the results mentioned blow have been already known mathematically, but their interpretation might be important benchmarks for the following subsections in which we will examine how stochastic time changes effect on the stock index markets.

The bivariate characteristic function and the cumulant-generating function (3.2) and (3.4) in this case are given by

$$\Psi_t(\theta_1, \theta_2) = \exp\{\lambda(\theta_1)t + i\theta_2\}, \qquad \text{and} \qquad \psi_t(\theta_1, \theta_2) = \lambda(-i\theta_1)t + \theta_2,$$

respectively. Then, by Propositions 1 and 2, the expected rate of return on the stock index and the interest rate in market equilibrium have the forms $\mu(t) = \mu t$ and $r(t) = rt$ for all $t \geq 0$, respectively, where

$$\mu = \frac{\delta}{1-a} - \frac{\varphi_Y(ia-i)}{1-a} + \varphi_Y(-i), \quad \text{and} \quad r = \frac{\delta}{1-a} - \frac{a}{1-a}\varphi_Y(ia-i) - \varphi_Y(ia). \tag{4.1}$$

The above results show that the expected rate of return on the stock index and the interest rate do not have any term structures in this model. The equity risk premium in a unit period is given by

$$\mu - r = -\varphi_Y(ia - i) + \varphi_Y(ia) + \varphi_Y(-i).$$

The characteristic function of the physical distribution of a log-return on the stock index can be represented as

$$\Phi^{\mathbb{P}}_{R_t}(\theta) = \exp\left\{t\left[i\theta(\mu - \varphi_Y(-i)) + \varphi_Y(\theta)\right]\right\},$$

while following to Propositions 2 and 3, the characteristic function of the risk-neutral distribution is given by

$$\Phi^{\mathbb{Q}}_{R_t}(\theta) = \exp\left\{t\left[i\theta(r - \varphi^{\mathbb{Q}}_Y(-i)) + \varphi^{\mathbb{Q}}_Y(\theta)\right]\right\}.$$

Here, $\varphi^{\mathbb{Q}}_Y$ denotes the characteristic exponent of the Lévy process $(Y_t)_{t\geq 0}$ under the risk-neutral measure $\mathbb{Q}$ given by

$$\begin{aligned}\varphi^{\mathbb{Q}}_Y(\theta) &= \varphi_Y(\theta + ia) - \varphi_Y(ia)\\ &= i\tilde{\alpha}\theta - \frac{1}{2}\beta\theta^2 + \int_{-\infty}^{\infty}\left(e^{i\theta y} - 1 - i\theta y\mathbf{1}_{\{|y|\leq 1\}}\right)\tilde{\nu}_Y(dy),\end{aligned}$$

where

$$\tilde{\alpha} := \alpha - \beta a + \int_{-1}^{1} y\left(e^{-ay} - 1\right)\nu_Y(dy), \quad \text{and} \quad \tilde{\nu}_Y(y) := e^{-ay}\nu_Y(y).$$

Using these results, the physical and risk-neutral cumulants of the stock index log-return can be easily obtained. It is obvious that the projection of the pricing kernel defined in (3.17) is monotonically decreasing function of the log-return of the stock index.

Next, due to the properties that Lévy processes have stationary and independent increments, the derivation of the expected realized variance of the stock index is straightforward. That is,

$$\mathbb{E}^{\mathbb{P}}\left[V_T\right] = Tc^{\mathbb{P}}_2(R_1) + \frac{T^2}{L}c^{\mathbb{P}}_1(R_1)^2, \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}\left[V_T\right] = Tc^{\mathbb{Q}}_2(R_1) + \frac{T^2}{L}c^{\mathbb{Q}}_1(R_1)^2.$$

The second terms on the right sides of the above two equations are discretization errors of the expected realized variance and converge to zero when $L \to \infty$.

4.1 BS Model

Consider the Black-Scholes model (Black & Scholes, 1973), that is,

$$\varphi_Y(\theta) = -\frac{1}{2}\sigma^2\theta^2, \tag{4.2}$$

where σ is a positive constant. In this model, the expected rate of return and the interest rate in (4.1) are represented as

$$\mu = \frac{\delta}{1-a} + \frac{1}{2}\sigma^2 a, \quad \text{and} \quad r = \frac{\delta}{1-a} - \frac{1}{2}\sigma^2 a,$$

respectively, and the equity risk premium is $\mu - r = \sigma^2 a$. Namely, if a is positive, the equity risk premium is positive. This is plausible. However, the following results contradict the stylized facts observed in stock index markets. The physical and risk-neutral variance of the log-return are the same, that is, $c_2^{\mathbb{P}}(R_1) = c_2^{\mathbb{Q}}(R_1) = \sigma^2$, and the skewness and excess kurtosis under both the physical and risk-neutral measure are zero. Furthermore, the expected realized variance under the physical measure exceeds that under the risk-neutral measure when the equity risk premium is positive because the first physical cumulant is larger than the first risk-neutral cumulant, which are parts of the discretization error of the expected realized variance.

4.2 VG Model

Consider the variance gamma model (Madan & Seneta, 1990) with zero skewness under the physical measure, that is,

$$\varphi_Y(\theta) = -\frac{1}{\kappa}\log\left(1 + \frac{1}{2}\kappa\sigma^2\theta^2\right), \tag{4.3}$$

where σ and κ are positive constants such that $\kappa\sigma^2 a^2 < 2$. In this model, we have

$$c_2^{\mathbb{P}}(R_1) = \sigma^2, \qquad c_3^{\mathbb{P}}(R_1) = 0, \qquad c_4^{\mathbb{P}}(R_1) = 3\sigma^4\kappa.$$

Under the physical measure, this model has zero skewness, but a positive excess kurtosis. On the other hand, the characteristic exponent under the risk-neutral measure has the form

$$\varphi_Y^{\mathbb{Q}}(\theta) = -\frac{1}{\kappa}\log\left(1 + \frac{1}{2}\kappa\tilde{\sigma}^2\theta^2 - i\omega\kappa\theta\right),$$

where

$$\tilde{\sigma} := \frac{\sigma}{\sqrt{1 - \frac{1}{2}\kappa\sigma^2 a^2}}, \qquad \text{and} \qquad \omega := -\frac{\sigma^2 a}{\sqrt{1 - \frac{1}{2}\kappa\sigma^2 a^2}}.$$

Recall that $\omega \in \mathbf{R}$ is skewness parameter of the variance gamma model. If the correlation between the stock index return and the aggregate consumption is positive, that is to say $a > 0$, skewness parameter is negative and the risk-neutral distribution of the log-return on the stock index has negative skewness. More precisely, we have

$$c_2^{\mathbb{Q}}(R_1) = \tilde{\sigma}^2 + \omega^2\kappa, \qquad c_3^{\mathbb{Q}}(R_1) = 3\tilde{\sigma}^2\omega\kappa + 2\omega^3\kappa^2, \qquad c_4^{\mathbb{Q}}(R_1) = 3\tilde{\sigma}^4\kappa + 6\omega^4\kappa^3 + 12\tilde{\sigma}^2\omega^2\kappa^2.$$

As a result, the variance gamma model defined in (4.3) satisfies the stylized facts (3.14), (3.15), and (3.16) when a is positive. Moreover, with sufficient frequency of monitoring dates such that the discretization errors is small enough, the stylized fact that the expected realized variance under the risk-neutral measure exceeds that under the physical measure, that is, the inequality (3.24), is also satisfied. Regrettably, the projection of the pricing kernel (3.17) in this model is monotonically decreasing in the log-return. Thereby, this model cannot generate any negative expected returns of call options.

4.3 NIG Model

Consider the normal inverse Gaussian model (Barndorff-Nielsen, 1997) with zero skewness under the physical measure defined as

$$\varphi_Y(\theta) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 + \kappa\sigma^2\theta^2}, \tag{4.4}$$

where σ and κ are positive constants such that $\kappa\sigma^2 a^2 < 1$, and

$$c_2^{\mathbb{P}}(R_1) = \sigma^2, \qquad c_3^{\mathbb{P}}(R_1) = 0, \qquad c_4^{\mathbb{P}}(R_1) = 3\sigma^4\kappa.$$

Under the physical measure, this model also has zero skewness, but a positive excess kurtosis. The characteristic exponent of this model under the risk-neutral measure has the form

$$\varphi_Y^{\mathbb{Q}}(\theta) = \frac{1}{\tilde{\kappa}} - \frac{1}{\tilde{\kappa}}\sqrt{1 + \tilde{\kappa}\tilde{\sigma}^2\theta^2 - 2i\omega\tilde{\kappa}\theta},$$

where

$$\tilde{\kappa} := \frac{\kappa}{\sqrt{1-\kappa\sigma^2 a^2}}, \qquad \tilde{\sigma} := \frac{\sigma}{\sqrt[4]{1-\kappa\sigma^2 a^2}}, \qquad \text{and} \qquad \omega := -\frac{\sigma^2 a}{\sqrt{1-\kappa\sigma^2 a^2}}.$$

Recall that ω is skewness parameter of the normal inverse Gaussian model. We obtain

$$c_2^{\mathbb{Q}}(R_1) = \tilde{\sigma}^2 + \omega^2\tilde{\kappa}, \quad c_3^{\mathbb{Q}}(R_1) = 3\tilde{\sigma}^2\omega\tilde{\kappa} + 3\omega^3\tilde{\kappa}^2, \quad c_4^{\mathbb{Q}}(R_1) = 3\tilde{\sigma}^4\tilde{\kappa} + 15\omega^4\tilde{\kappa}^3 + 18\tilde{\sigma}^2\omega^2\tilde{\kappa}^2.$$

As a result, we can mention the same remarks about capability explaining the stylized facts in the normal Gaussian model as in the variance gamma model.

5 Stochastic Activity Rate

In this section, we consider the time-changed Lévy processes such that the characteristic function of the joint distribution of $(\int_t^T v_{s-}ds, v_T)$ conditional on $\mathcal{F}_t$ under $\mathbb{P}(\theta)$ has an exponential affine form. That is, we assume

$$\mathbb{E}_t^{\mathbb{P}(\theta)}\left[\exp\left\{\xi_1\int_t^T v_{s-}ds + \xi_2 v_T\right\}\right] = \exp\left\{A_{T-t}^{\theta}(\xi_1,\xi_2) + B_{T-t}^{\theta}(\xi_1,\xi_2)v_t\right\}, \tag{5.1}$$

is well defined for both complex and real arguments on some region $\mathcal{D} \subset \mathbf{C}^2$, where A_{T-t}^{θ} and B_{T-t}^{θ} are some deterministic functions defined on $\mathcal{D}$. If an analytically tractable time-changed Lévy process is given, then these functions can be expressed in closed-form. In this case, from Lemma 2, we have

$$\psi_t(\theta_1,\theta_2) = A_t^{-i\theta_1}(\lambda(-i\theta_1),\theta_2) + B_t^{-i\theta_1}(\lambda(-i\theta_1),\theta_2). \tag{5.2}$$

Therefore, plugging (5.2) into (3.6), (3.7), and (3.8), the expected rate of return on the stock index, the interest rate, and the equity risk premium can be represented in terms of the functions A_t^{θ} and B_t^{θ}. They determine the term structures of the expected returns and the interest rates in market equilibrium. Recall that the proper Lévy processes discussed in Section 4 can only generate the flat term structures. Furthermore, the cumulant generating functions of the physical and the risk-neutral distributions of the log-return on the stock index (3.14) and (3.14) can be written as

$$\phi_{R_t}^{\mathbb{P}}(\theta) = \theta\mu(t) + A_t^{-i\theta}(\lambda(-i\theta),0) + B_t^{-i\theta}(\lambda(-i\theta),0),$$

and

$$\phi_{R_t}^{\mathbb{Q}}(\theta) = \theta\mu(t) + A_t^{i[a-\theta]}(\lambda(i[a-\theta]),-b) + B_t^{i[a-\theta]}(\lambda(i[a-\theta]),-b) - \psi_t^0,$$

respectively. Here, we define $\psi_t^0 := A_t^{ia}(\lambda(ia), -b) + B_t^{ia}(\lambda(ia), -b)$, which is just a time dependent value corresponding to the logarithm of the time-t normalization factor defined in Proposition 3. Finally, applying Proposition 5, we obtain the representations of the cumulant generating functions of the log-return increment as

$$\phi^{\mathbb{P}}_{\Delta R_l}(\theta) = A_{\Delta t}^{-i\theta}(\lambda(-i\theta), 0) + A^0_{t_{l-1}}(0, B_{\Delta t}^{-i\theta}(\lambda(-i\theta), 0)) + B^0_{t_{l-1}}(0, B_{\Delta t}^{-i\theta}(\lambda(-i\theta), 0)),$$

and

$$\begin{aligned}\phi^{\mathbb{Q}}_{\Delta R_l}(\theta) = \theta\left(\mu(t_l) - \mu(t_{l-1})\right) + A_{\Delta t}^{i[a-\theta]}(\lambda(i[a-\theta]), -b) + A^{ia}_{t_{l-1}}(\lambda(ia), B_{\Delta t}^{i[a-\theta]}(\lambda(i[a-\theta]), -b))\\ + B^{ia}_{t_{l-1}}(\lambda(ia), B_{\Delta t}^{i[a-\theta]}(\lambda(i[a-\theta]), -b)) - \psi^0_{t_l}.\end{aligned}$$

After all, our purpose in this section is to provide the explicit representations of the functions A^θ_{T-t} and B^θ_{T-t} when a specific activity rate $(v_t)_{t\geq 0}$ is given. Fortunately, such activity rate processes have been already known in past literature. As an example, we will treat the square root process and the non-Gaussian OU process, which are most admissible activity rate for the time-changed Lévy processes.

5.1 Square Root Process

Suppose that the activity rate process follows the square root process under $\mathbb{P}(\theta)$.

$$dv_t = (k_0 + k_1 v_t)dt + c\sqrt{v_t}dW_t^\theta, \qquad t \geq 0, \tag{5.3}$$

with $v_0 = 1$, where $k_0, c \in \mathbf{R}^+$, $k_1 \in \mathbf{C}$, and $(W_t^\theta)_{t\geq 0}$ is a Brownian motion under $\mathbb{P}(\theta)$. The square root process defined in (5.3) is well known as the CIR process (Cox et al., 1985) in the field of interest rate modeling. Note that the activity rate (5.3) might take values in $\mathbf{C}$ under $\mathbb{P}(\theta)$. In this case, the functions A^θ_{T-t} and B^θ_{T-t} are given by

$$A^\theta_{T-t}(\xi_1, \xi_2) = -\frac{2k_0}{c^2}\log\left|\Upsilon_1(T-t, \xi_1, \xi_2)\right|, \quad \text{and} \quad B^\theta_{T-t}(\xi_1, \xi_2) = \frac{2}{c^2}\frac{\Upsilon_2(T-t, \xi_1, \xi_2)}{\Upsilon_1(T-t, \xi_1, \xi_2)}. \tag{5.4}$$

Here, we define

$$\Upsilon_1(t, \xi_1, \xi_2) = \begin{cases} \left(-\frac{1}{2}(k_1 + c^2\xi_2)t + 1\right)e^{\frac{1}{2}k_1 t}, & \text{if } k_1^2 - 2c^2\xi_1 = 0, \\ \frac{C_1}{d_+}e^{-d_+ t} - \frac{C_1}{d_-}e^{-d_- t}, & \text{otherwise,} \end{cases}$$

$$\Upsilon_2(t, \xi_1, \xi_2) = -\frac{\partial}{\partial t}\Upsilon_1(t, \xi_1, \xi_2),$$

where $C_1 = d_+(c^2\xi_2/2 - d_-)(2d_0)^{-1}$, $C_2 = C_1 - c^2\xi_2/2$, $d_\pm = -k_1/2 \pm d_0$, $d_0 = \frac{1}{2}\sqrt{k_1^2 - 2c^2\xi_1}$. The derivation of the functions above can be found in the appendix of Umezawa & Yamazaki (2014) for instance.

5.1.1 Heston Model

Heston's stochastic volatility model (Heston, 1993) is defined by the SDE

$$dY_t = \sigma dW_t^1, \tag{5.5}$$

$$dv_t = k(1 - v_t)dt + c\sqrt{v_t}dW_t^2, \tag{5.6}$$

where $(W_t^1)_{t\geq 0}$ and $(W_t^2)_{t\geq 0}$ are Brownian motions with $dW_t^1 dW_t^2 = \rho dt$, and $\sigma, k, c > 0$ and $\rho \in (-1,1)$ are parameters with the Feller condition $2k > c^2$ ensuring that the activity rate remains positive. This is only the case in this paper that the log-return process is correlated with the activity rate process. The negative skewness of the physical distribution of the log return on the stock index can be accommodated by negatively correlating Y_t and v_t, that is, $\rho < 0$. The activity rate (5.6) is normalized by setting the initial value and the mean-reverting level to be one. This normalization makes it easy to compare the Heston model with other time-changed Lévy models. One can confirm that, by putting $V_t = \sigma^2 v_t$, the normalized representation (5.6) is converted into the standard formulation of the Heston model in which both the initial value of the variance and the mean-reverting level are equal to σ^2.

Define the measure $\mathbb{P}(\theta)$ as

$$Z_t(\theta) = \exp\left\{ i\theta\sigma \int_0^t \sqrt{v_s} dW_s^1 + \frac{1}{2}\theta^2\sigma^2 \int_0^t v_s ds \right\}.$$

Applying Girsanov's theorem, we obtain the activity rate process under $\mathbb{P}(\theta)$ as follows.

$$dv_t = (k - [k - i\theta\sigma c\rho]v_t)dt + c\sqrt{v_t} dW_t^\theta, \tag{5.7}$$

where $W_t^\theta := W_t^2 - i\theta\sigma\rho \int_0^t \sqrt{v_s} ds$ is a Brownian motion under $\mathbb{P}(\theta)$. Therefore, in the Heston model, we set $k_0 = k$ and $k_1 = i\theta\sigma c\rho - k$ in (5.3) with the characteristic exponent (4.2).

5.1.2 Other Models

We can immediately implement various types of time-changed Lévy processes with the CIR type activity rate other than the Heston model. And yet almost all the models in past literature have had no correlation between the log-return and the activity rate. For example, the VG-CIR model is a time-changed Lévy process having the variance gamma process with the activity rate (5.6), while the NIG-CIR model is composed of the normal inverse Gaussian process with the activity rate (5.6). In these models, the underlying Lévy processes have been assumed to be independent of the activity rates. If the zero skew variance gamma model (4.3) is adopted as the underlying Lévy process, the generated VG-CIR model is distributed with zero skewness under the physical measure. This is rather favorable to reconcile the stylized fact observed in time series data of log-returns. In the independent case, the change of measure from $\mathbb{P}$ to $\mathbb{P}(\theta)$ is unnecessary and we have only to put $k_0 = k$ and $k_1 = -k$ in (5.3) to use the formula (5.4).

5.2 Non-Gaussian OU Process

Suppose that the activity rate process follows the non-Gaussian OU process (Barndorff-Nielsen and Shephard, 2001) under $\mathbb{P}$.

$$dv_t = -kv_t dt + dL_{kt}, \qquad t \geq 0, \tag{5.8}$$

with $v_0 = 1$, where k is a positive constant and $(L_t)_{t\geq 0}$ is a non-decreasing Lévy process. The stochastic process $(L_t)_{t\geq}$ is called the *background driving Lévy process* (hereafter BDLP) of the activity rate (5.8). The strong solution to the SDE (5.8) is the form

$$v_t = e^{-kt} + \int_0^t e^{-k(t-s)} dL_{ks}.$$

Since the BDLP $(L_t)_{t\geq}$ is non-decreasing, the activity rate v_t is strictly positive for every $t \geq 0$ and bounded from below by the deterministic function e^{-kt}. Although it is certainly possible to incorporate the correlation between the log-return on the stock index and its activity rate by a slightly modification in this model, see Yamazaki (2016) for example, this study assumes that they are independent of each other. Under the independence assumption, we can set $\theta = 0$ in (5.1) and the functions A^0_{T-t} and B^0_{T-t} are given by

$$A^0_{T-t}(\xi_1, \xi_2) = k \int_t^T \varphi_L(B^0_{T-s}(-i\xi_1, -i\xi_2))ds, \tag{5.9}$$

$$B^0_{T-t}(\xi_1, \xi_2) = \xi_1 \frac{1 - e^{-k(T-t)}}{k} + \xi_2 e^{-k(T-t)}, \tag{5.10}$$

where φ_L is the characteristic exponent of the BDLP $(L_t)_{t\geq 0}$. The derivation of (5.9) and (5.10) can be found in the proposition 1 of Yamazaki (2016). As will shown in the following subsections, the integral on the right side of (5.9) has a closed-form expression in some cases.

5.2.1 Γ-OU Process

It is known that, when a one-dimensional distribution D is given, there exists an OU process whose stationary distribution is D if and only if D is self-decomposable (see the section 17 in Sato (1999) for example). Moreover, Barndorff-Nielsen and Shephard (2001) demonstrated that, given the cumulant generating function of an arbitrary self-decomposable distribution with positive support, the characteristic function of the BDLP of the corresponding OU process can be specified immediately.

Here, we consider the activity rate known as the Γ-OU process, which has the gamma distribution as the stationary distribution with positive support in (5.8). The gamma distribution $\Gamma(p, q)$ is self-decomposable, where $p > 0$ is shape parameter and q is rate parameter. That is, the cumulant generating function of the gamma distribution is written as

$$\phi_\Gamma(\theta) = -p \log \left(1 - \frac{i\theta}{q}\right).$$

In this case, the characteristic exponent of the corresponding BDLP is given by

$$\varphi_L(\theta) = \frac{ip\theta}{q - i\theta}. \tag{5.11}$$

Since the Lévy measure of the BDLP has the form

$$\nu_L(dy) = pqe^{-qy}dy,$$

and $\nu_L(\mathbf{R}^+) < +\infty$, the Γ-OU process $(v_t)_{t\geq 0}$ has a finite number of positive jumps in any time periods. The closed-form expression of (5.9) is given by

$$A^0_{T-t}(\xi_1, \xi_2) = \frac{p}{a_1 + iq} \left\{ iq \log \left(\frac{a_1 + a_2 e^{-k(T-t)} + iq}{a_1 + a_2 + iq} \right) - a_1 k(T-t) \right\},$$

where $a_1 = -ik^{-1}\xi_1$ and $a_2 = i(k^{-1}\xi_1 - \xi_2)$.

5.2.2 IG-OU Process

Next, we consider the activity rate called the IG-OU process whose stationary distribution is the inverse Gaussian distribution in (5.8). The inverse Gaussian distribution $\mathrm{IG}(p, q)$ is also self-decomposable. Here, $p, q > 0$ are parameters by the Barndorff-Nielsen (1997)'s parameterization. The cumulant generating function of the normal inverse Gaussian distribution is written as

$$\phi_{\mathrm{IG}}(\theta) = pq - p\sqrt{q^2 - 2i\theta}.$$

In this case, the characteristic exponent of the corresponding BDLP is given by

$$\varphi_L(\theta) = \frac{ip\theta}{\sqrt{q^2 - 2i\theta}}. \tag{5.12}$$

Since the Lévy measure of the BDLP has the form

$$\nu_L(dy) = \frac{p}{2\sqrt{2\pi}} y^{-\frac{3}{2}}(1 + q^2 y)e^{-\frac{1}{2}q^2 y} dy,$$

and $\nu_L(\mathbf{R}^+) = +\infty$, the IG-OU process jumps infinitely often in any finite time intervals. The closed-form expression of (5.9) is given by

$$\begin{aligned} A^0_{T-t}(\xi_1, \xi_2) = p\Bigg\{ & \sqrt{a_3 - 2ia_2 e^{-k(T-t)}} - \sqrt{a_3 - 2ia_2} \\ & + \frac{2ia_1}{\sqrt{a_3}} \log\left(\frac{\sqrt{a_3 - 2ia_2 e^{-k(T-t)}} + \sqrt{a_3}}{\sqrt{a_3 - 2ia_2} + \sqrt{a_3}} \right) + \frac{ia_1 k}{\sqrt{a_3}}(T - t) \Bigg\}, \end{aligned}$$

where $a_1 = -ik^{-1}\xi_1$, $a_2 = i(k^{-1}\xi_1 - \xi_2)$, and $a_3 = q^2 - 2k^{-1}\xi_1$.

6 Numerical Examples

In this section, we implement two dynamic equilibrium models driven by time-changed Lévy processes to examine implications in stock index markets. One of them is the Heston model, in which the level of a stock index and its volatility are governed by continuous processes. This model can show the correlation between the stock index and its volatility under physical measure. Another example is the NIG Γ-OU model, which is generated by discontinuous jump processes with no correlation.

6.1 Heston Model

Suppose that the physical dynamics of a stock index follows the Heston model defined in Section 5.1.1. First, we treat the uncorrelated case. We then investigate the negatively correlated Heston model.

6.1.1 Uncorrelated Case

The parameters of the Heston model under a physical measure $\mathbb{P}$ are given in Panel A of Table 1. Note that correlation parameter ρ is set to be zero. That is, the leverage effect in the physical measure is absent. We assume that the annual equity risk premium of the stock index is 7.5% (i.e., $\mu(1) - r(1) = 0.075$) and the interest rate with maturity $T = 1$ is 2.5% (i.e., $r(1) = 0.025$). Panel B of Table 1 displays the results of the simple calibration exercise mentioned below. Given the Heston parameters in Panel A, four different scenarios are considered for macroeconomic parameter a. Recall that $a := \hat{a}\gamma$. As shown in Panel B, each scenario corresponds to an economy with distinct level of the representative investor's risk γ and/or regression coefficient $\hat{a}$. For a fixed level of parameter a, one can adjust the level of parameter b so that the annual equity risk premium is fixed at 7.5% by using equation (3.8) in Corollary 1. Next, for fixed levels of a and b, one can calibrate the level of the rate of time preference δ such that the interest rate with $T = 1$ is fixed at 2.5% by using equation (3.7) in Proposition 2. Note that all the calibrated levels of parameter b are negative. This means that increasing the volatility of the stock index deteriorates the aggregate consumption and is consistent with our intuition.

Table 2 shows the standard deviation (Stdv), skewness (Skew), and excess kurtosis (Kurt) of the distributions of log-return on the stock index under the physical measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$ for each scenario. In addition, the square root of the expected realized variance (RV) of the stock index with daily monitoring $\Delta t = 1/250$ is displayed. We stress that the physical distribution is not involved in macroeconomic parameters a, b, and δ. Panel A of Table 2 listing the statistics for the annual log-return on the stock index shows that the values of standard deviation are higher under the risk-neutral measures than under the physical measure. Moreover, they are decreasing in parameters a and b. The same tendency can be seen in the values of expected realized variance. The values of excess kurtosis are also higher under the risk-neutral measures than under the physical measure. However, risk-neutral kurtosis is not necessarily monotonic with respect to parameters a and b. The values of skewness are negatively larger under the risk-neutral measures than under the physical measure. That is, risk-neutral skewness is obviously negative, whereas physical skewness is nearly zero. This is plausible. The values of risk-neutral skewness are slightly decreasing when parameters a and b are increasing. Panel B of Table 2 displaying the semi-annual log return on the stock index shows the same tendency as Panel A except for the excess kurtosis. Panel B of Table 2 shows smaller values of risk-neutral kurtosis in Cases 1 and 3 than physical kurtosis.

Table 1: Model parameters

Panel A: Heston parameters

σ	k	c	ρ
0.125	0.800	0.800	0.000

Panel B: Macroeconomic parameters

	a	b	δ
Case 1	3.3000	-1.6285	0.4785
Case 2	3.5000	-1.4315	0.2977
Case 3	3.7000	-1.2244	0.1448
Case 4	3.9000	-1.0060	0.0184

Figures 1 and 2 depict the Black-Scholes implied volatilities against moneyness K/S_0 with maturities $T = 1$ and 0.5, respectively. Both figures show the shape of *volatility skew*. That is, the implied volatilities are monotonically decreasing in moneyness. This implies that the risk-averse representative investor has to pay relatively high premiums for deep OTM put options even though the stock index is not correlated with its volatility under the physical measure. Figures 3 and 4 plot the pricing kernels against log-returns R_T with maturities $T = 1$ and 0.5, respectively. As shown in Figures 3 and 4, Cases 1, 2, and 3 support the U-shaped pricing kernels, whereas in Case 4, the pricing kernel is monotonically decreasing in the log-return. Obviously, a key element for depicting the U-shaped pricing kernels is the depth of negativity of parameter b. The more negative is parameter b, the steeper the slope of the U-shaped pricing kernels in the increasing region. Figures 5 and 6 draw the expected returns of call options on the stock index against moneyness K/S_0 with maturities $T = 1$ and 0.5, respectively. The

Table 2: Characteristic of distributions and realized variance

Panel A: Fundamental statistics for $T = 1$

	Stdv	RV	Skew	Kurt
$\mathbb{P}$ measure	0.1250	0.1250	-0.0229	0.3689
$\mathbb{Q}$ measure				
Case 1	0.1544	0.1538	-0.2207	0.4143
Case 2	0.1500	0.1493	-0.2235	0.4303
Case 3	0.1460	0.1450	-0.2259	0.4244
Case 4	0.1423	0.1409	-0.2279	0.4231

Panel B: Fundamental statistics for $T = 0.5$

	Stdv	RV	Skew	Kurt
$\mathbb{P}$ measure	0.0884	0.0884	-0.0106	0.2465
$\mathbb{Q}$ measure				
Case 1	0.0994	0.0993	-0.0914	0.1682
Case 2	0.0979	0.0978	-0.0945	0.2618
Case 3	0.0964	0.0962	-0.0974	0.2410
Case 4	0.0950	0.0947	-0.1001	0.2541

graphs of the expected returns are not smooth in the right-hand side of the figures because it is very difficult to compute deep OTM call options accurately. The expected returns in Cases 1, 2, and 3 have decreasing regions, whereas the expected returns in Case 4 are monotonically increasing. The more negative is parameter b, the steeper the downward slope of the expected call returns. Furthermore, a negative region of the expected returns is observed in Case 1. Therefore, this model can generate negative expected returns on deep OTM call options when parameter b is strongly negative. By observing Figures 3 and 4 with Figures 5 and 6, we can confirm that this example is consistent with Theorem 1 of Bakshi et al. (2010).

In conclusion, we perform reverse engineering of the stylized facts with which we are concerned. Notably, the dynamic equilibrium model reproduces Theorem 1 of Bakshi et al. (2010). However, the term structures of the equity risk premiums and interest rates have some problems. Figure 7 plots the equity risk premiums per unit time defined as $(\mu(t) - r(t))/t$. The term structure of the equity risk premiums draws bump curves. As shown in Figure 8, the yield to maturity defined as $r(t)/t$ has negative slope. Furthermore, it takes negative values in long maturities.

6.1.2 Correlated Case

Next, we consider the case of the stock index negatively correlated with its volatility under a physical measure $\mathbb{P}$. The Heston model parameters are listed in Panel A of Table 3. Panel B of Table 3 shows that while three different scenarios are considered for correlation parameter ρ, macroeconomic parameter a is fixed at 2.0 in all scenarios. As with the uncorrelated case, the respective levels of parameters b and δ are calibrated so that the annual equity risk premium and interest rate are fixed at 7.5% and 2.5%, respectively.

Table 4 displays the fundamental statistics of distributions of the log-return on the stock index and the square root of the expected realized variance with daily monitoring. The distinguishing traits are skewness and excess kurtosis. It is trivial that physical distributions have negative skewness depending on the correlation parameter. Remarkably, risk-neutral skewness becomes much more negative than physical skewness. Furthermore, the more negative

Figure 1: Implied Volatility in Uncorrelated Heston Model with $T = 1$

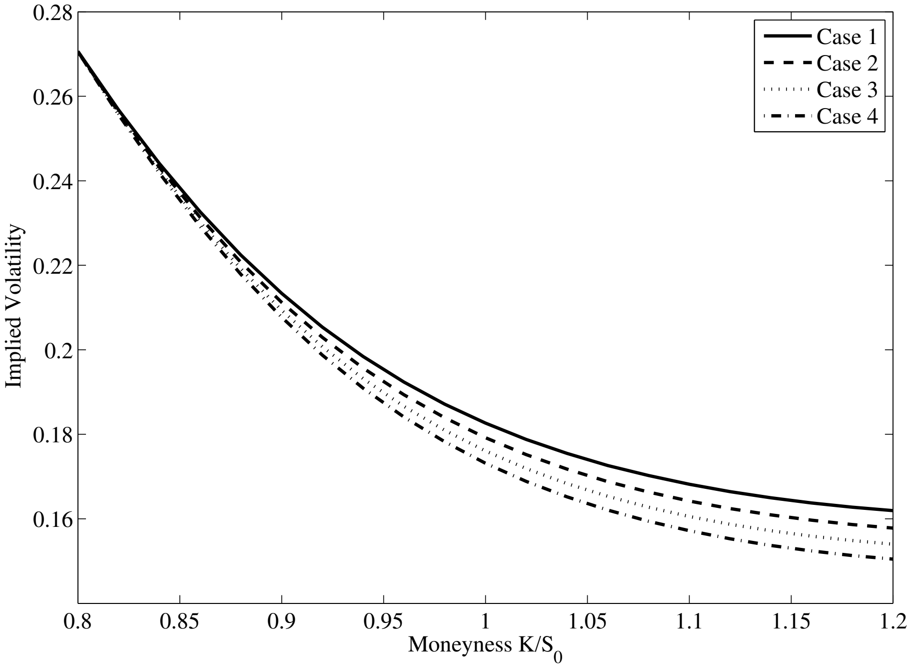


Figure 2: Implied Volatility in Uncorrelated Heston Model with $T = 0.5$

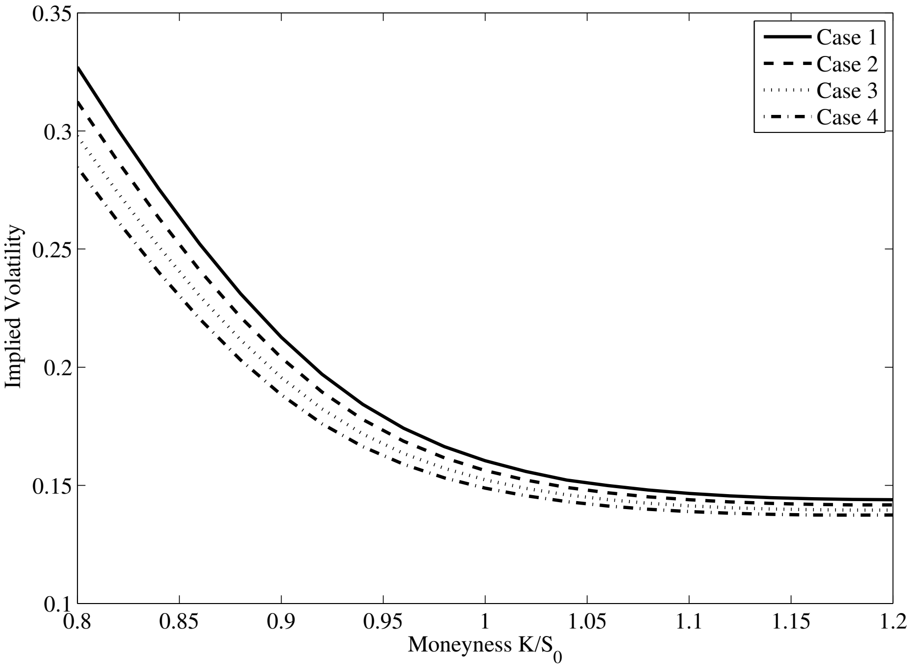


Figure 3: Pricing Kernel in Uncorrelated Heston Model with $T = 1$

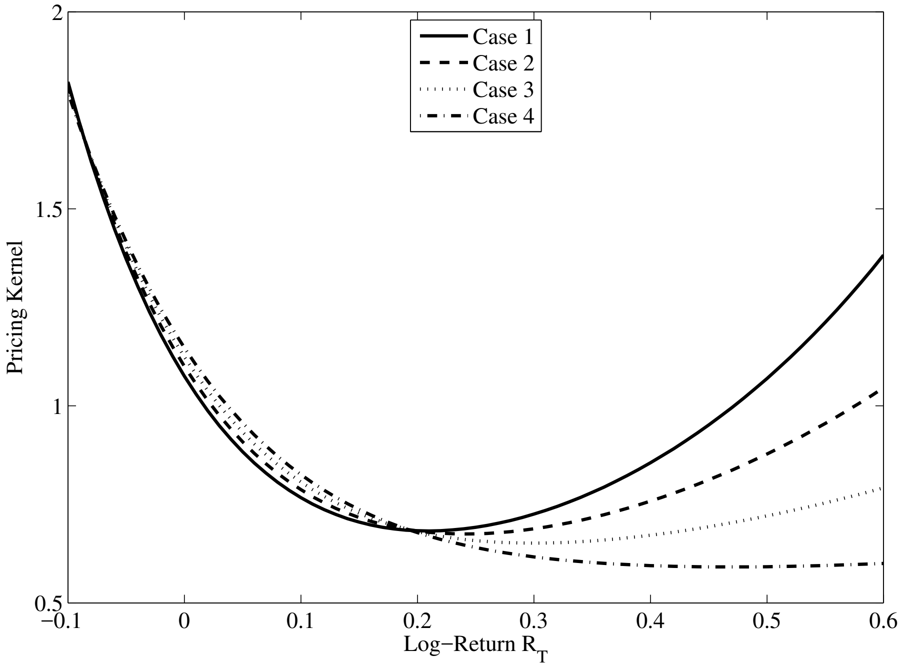


Figure 4: Pricing Kernel in Uncorrelated Heston Model with $T = 0.5$

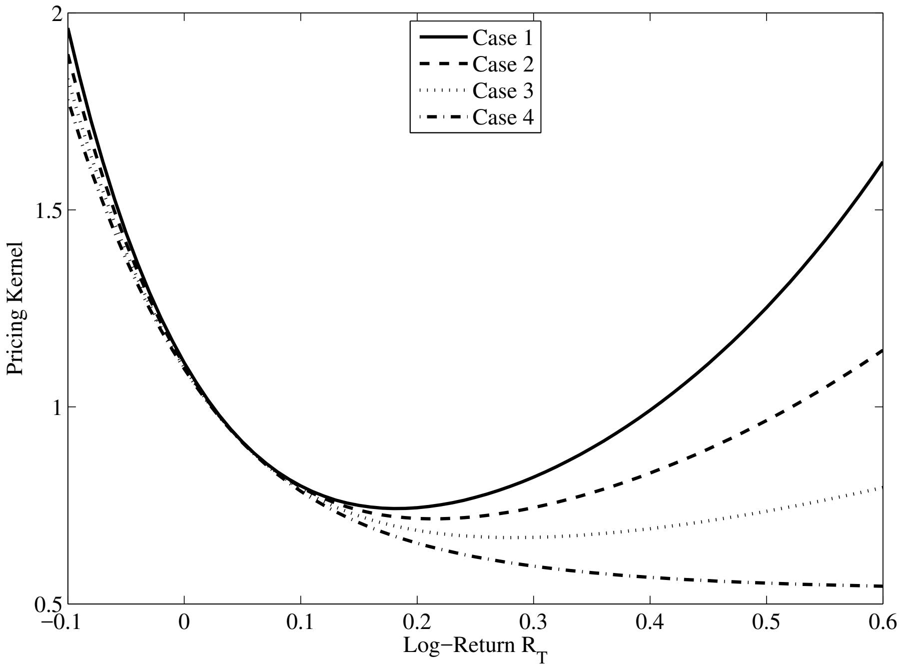


Figure 5: Expected Return of Call Option in Uncorrelated Heston Model with $T = 1$

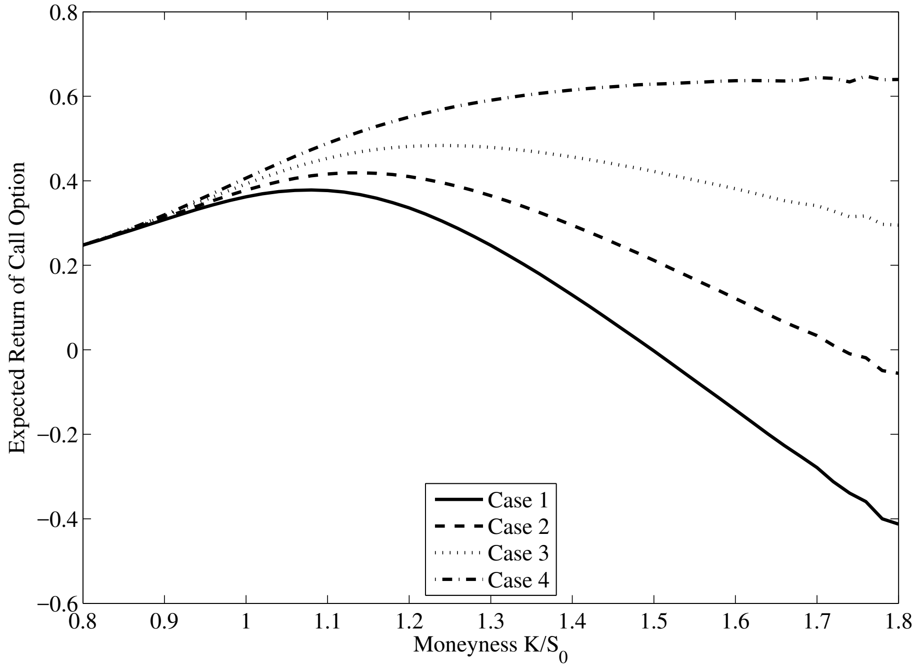


Figure 6: Expected Return of Call Option in Uncorrelated Heston Model with $T = 0.5$

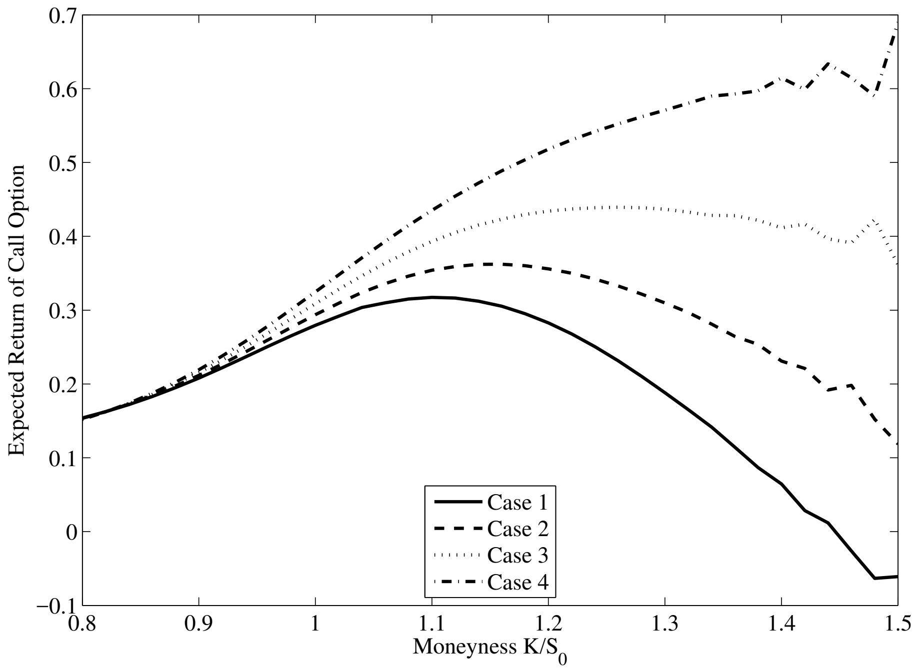


Figure 7: Equity Risk Premium in Uncorrelated Heston Model

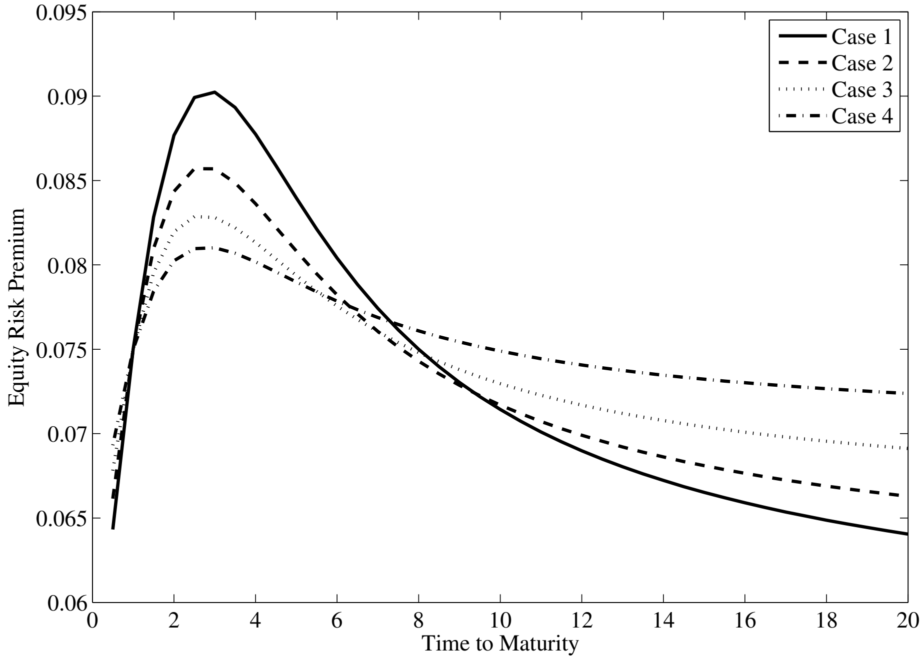
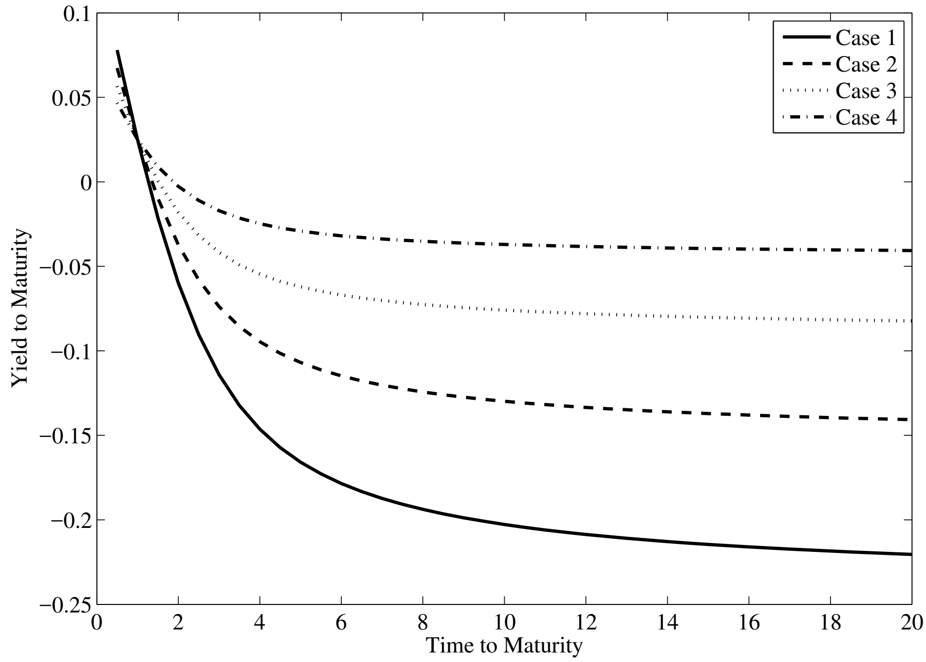


Figure 8: Yield to Maturity in Uncorrelated Heston Model



correlation parameter ρ, the more negative is the skewness and higher the kurtosis.

Table 3: Model parameters

Panel A: Heston parameters

σ	k	c	ρ
0.125	0.800	0.800	-0.200$\sim$-0.600

Panel B: Macroeconomic parameters

ρ	a	b	δ
-0.200	2.0000	-1.3816	0.3701
-0.400	2.0000	-0.9112	0.1144
-0.600	2.0000	-0.6668	0.0345

Table 4: Characteristic of distributions and realized variance

Panel A: Fundamental statistics for $T = 1$

ρ		Stdv	RV	Skew	Kurt
-0.200	$\mathbb{P}$ measure	0.1255	0.1250	-0.2086	0.4083
	$\mathbb{Q}$ measure	0.1553	0.1496	-0.3853	0.4911
-0.400	$\mathbb{P}$ measure	0.1260	0.1250	-0.3922	0.5166
	$\mathbb{Q}$ measure	0.1533	0.1415	-0.5341	0.6267
-0.600	$\mathbb{P}$ measure	0.1265	0.1250	-0.5737	0.6908
	$\mathbb{Q}$ measure	0.1559	0.1385	-0.6654	0.7737

Panel B: Fundamental statistics for $T = 0.5$

ρ		Stdv	RV	Skew	Kurt
-0.200	$\mathbb{P}$ measure	0.0886	0.0884	-0.1592	0.2649
	$\mathbb{Q}$ measure	0.0999	0.0980	-0.2359	0.1902
-0.400	$\mathbb{P}$ measure	0.0888	0.0884	-0.3068	0.3215
	$\mathbb{Q}$ measure	0.0993	0.0950	-0.3724	0.3371
-0.600	$\mathbb{P}$ measure	0.0890	0.0884	-0.4533	0.4341
	$\mathbb{Q}$ measure	0.1003	0.0938	-0.4957	0.4374

Figures 9 and 10 show that the Black-Scholes implied volatilities are volatility skew. As shown in Figure 9, when correlation parameter ρ is more negative, the slope of the implied volatility with maturity $T = 1$ is slightly steeper. This phenomenon gets the point. Conversely, Figure 10 demonstrates that the less negative the correlation parameter, the steeper is the slope of the implied volatility with maturity $T = 0.5$. This result might run counter to our intuition. Figures 11 and 12 depict the pricing kernels with $T = 1$ and 0.5, respectively. As shown in these figures, all the pricing kernels are U-shaped. The more negative the correlation parameter, the steeper is the slope of the pricing kernel in the increasing region. Figures 13 and 14 plot the expected returns of call options on the stock index with $T = 1$ and 0.5, respectively. The expected call returns have a decreasing region. We also observe negative call returns with $T = 1$ in the deep OTM region. The term structures of the equity risk premiums and the interest rates, which are omitted in this paper, draw shapes similar to Figures 7 and 8, respectively.

Figure 9: Implied Volatility in Correlated Heston Model with $T = 1$

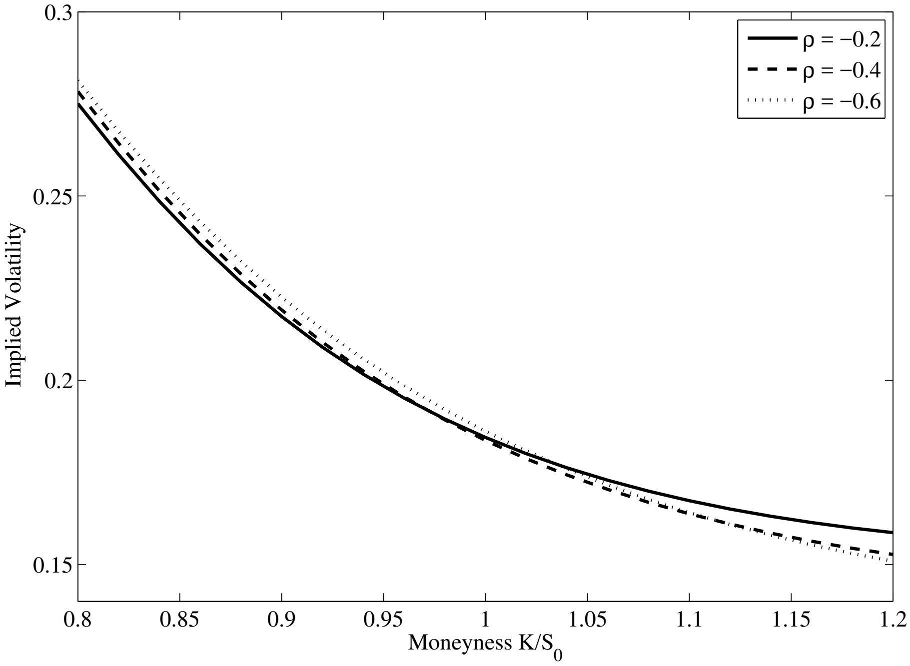


Figure 10: Implied Volatility in Correlated Heston Model with $T = 0.5$

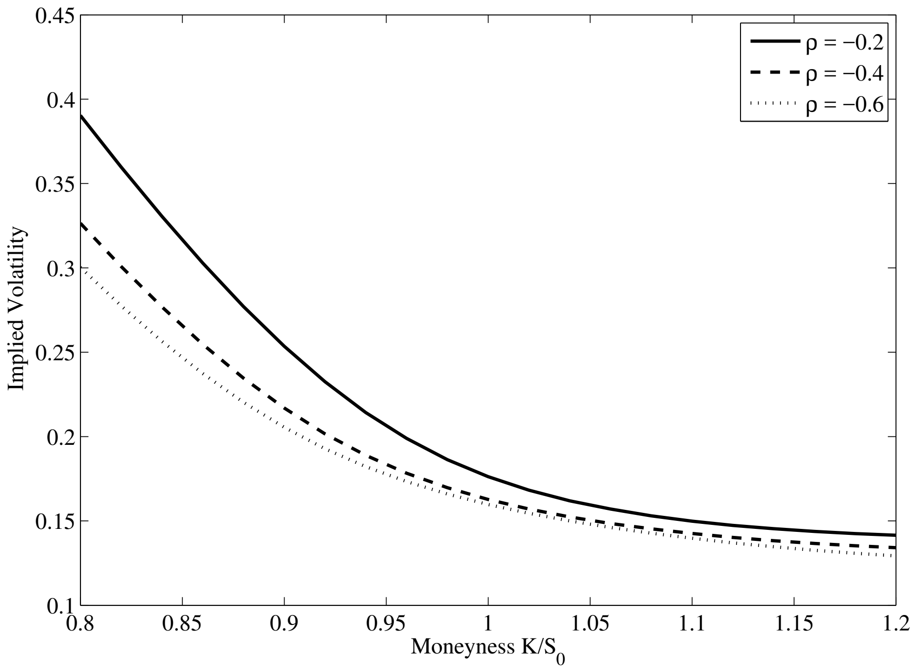


Figure 11: Pricing Kernel in Correlated Heston Model with $T = 1$

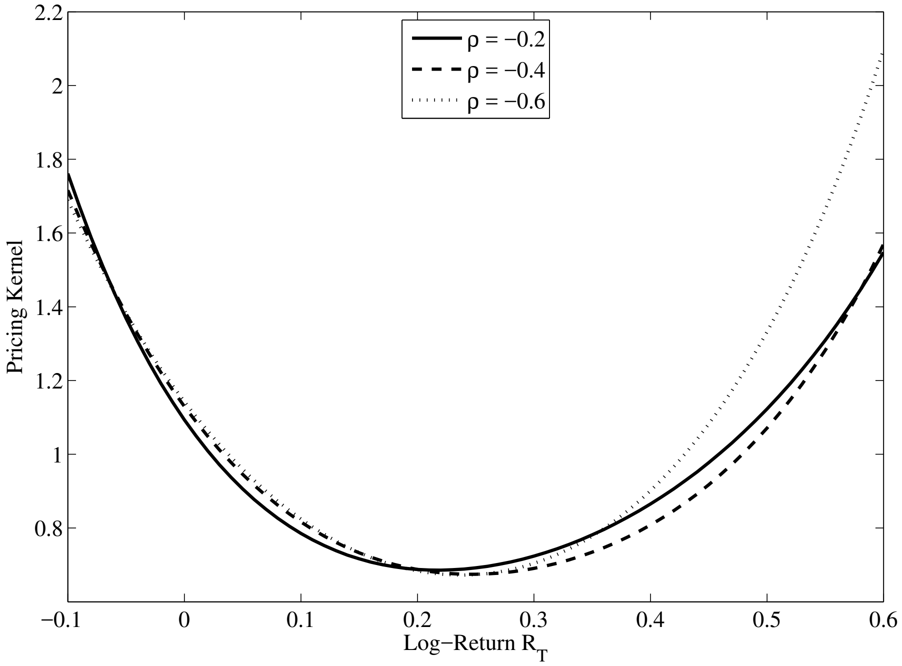


Figure 12: Pricing Kernel in Correlated Heston Model with $T = 0.5$

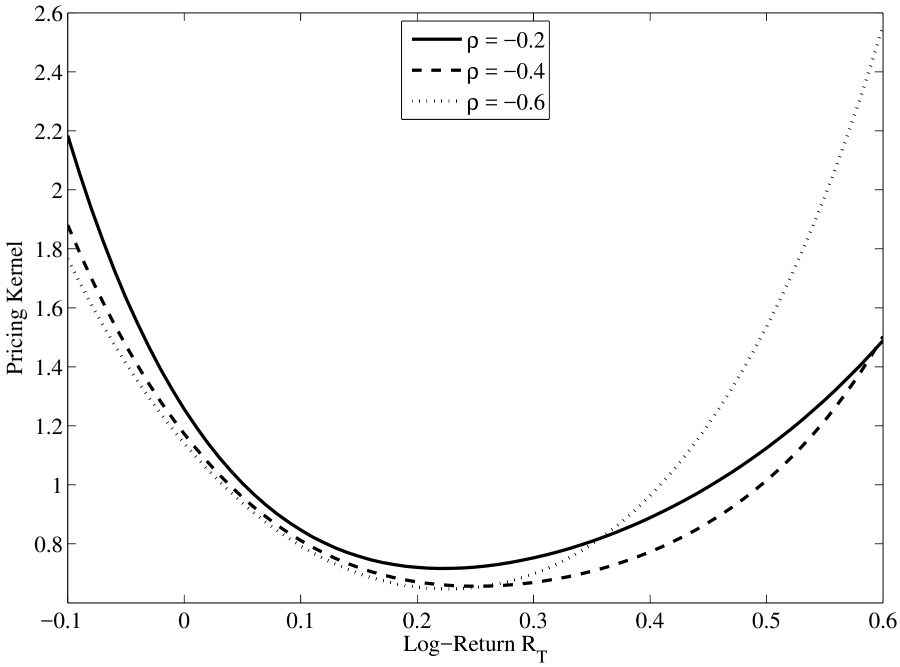


Figure 13: Expected Return of Call Option in Correlated Heston Model with $T = 1$

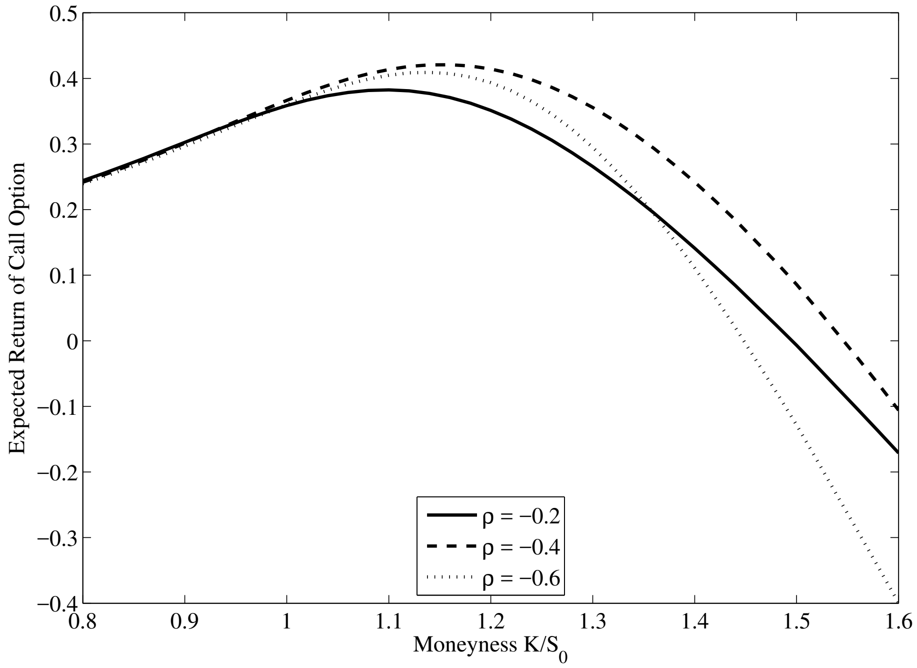
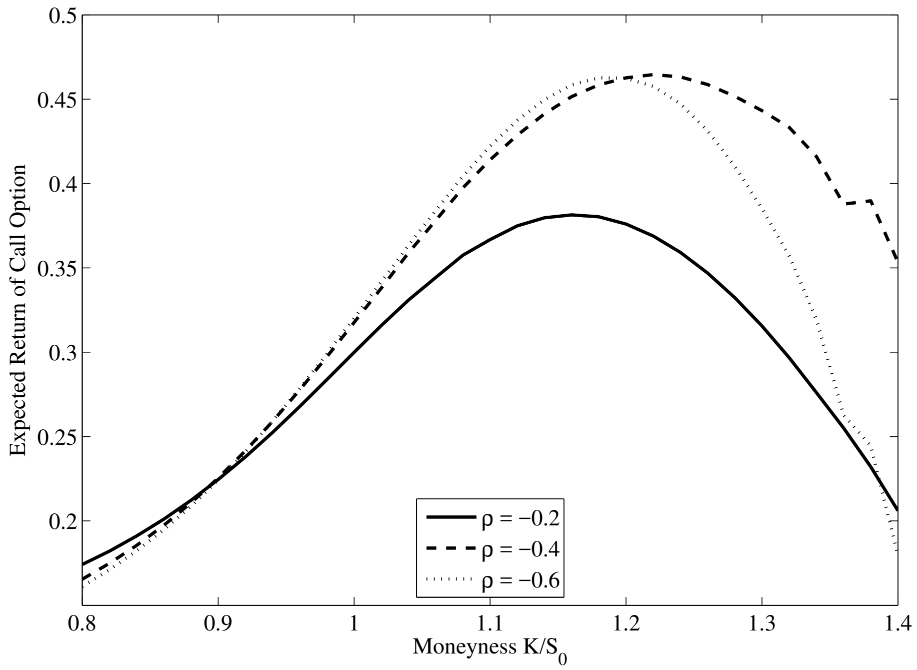


Figure 14: Expected Return of Call Option in Correlated Heston Model with $T = 0.5$



6.2 NIG Γ-OU Model

Suppose that the level of a stock index is driven by the NIG Γ-OU process that is a time-changed Lévy process composed of a normal inverse Gaussian process with the activity rate governed by the Γ-OU process. More concretely, the underlying Lévy process $(Y_t)_{t\geq 0}$ is defined by (4.4) and the activity rate process $(v_t)_{t\geq 0}$ is given by (5.8) and (5.11). The parameters of the NIG Γ-OU model under a physical measure $\mathbb{P}$ are listed in Panel A of Table 5. Note that the normal inverse Gaussian process $(Y_t)_{t\geq 0}$ has zero skewness and is independent of the activity rate process $(v_t)_{t\geq 0}$ under the physical measure. We consider four different scenarios for macroeconomic parameter a, which are the same as in Section 6.1.1. Then, fixing the annual equity risk premium at 7.5% and the interest rate at 2.5%, we adjust the level of parameters b and δ in the same manner as in Section 6.1.1. The calibration results are listed in Panel B of Table 5.

Table 6 shows the fundamental statistics of the return distributions. They have almost the same characteristic as under the uncorrelated Heston model in Section 6.1.1. The differences between the two models are as follows. Under risk-neutral probabilities, the square root of the expected realized variance and the excess kurtosis of the NIG Γ-OU model are larger than those of the Heston model. Risk-neutral skewness is more negative under the NIG Γ-OU model than under the Heston model.

Figures 15-20 depict the implied volatilities, the pricing kernels, and the expected returns of call options on the stock index under the NIG Γ-OU model. They are similar to those under the uncorrelated Heston model in Section 6.1.1. The distinctive characteristics of the NIG Γ-OU model are as follows. The volatility skew with $T = 0.5$ under the NIG Γ-OU model in Figure 16 is much steeper than that under the Heton model in Figure 2. The slope of the pricing kernel in the increasing region with $T = 0.5$ in Figure 18 is considerably steep, whereas the slope with $T = 1$ in Figure 17 is moderate. These results affect the shape of the graphs drawn by the expected returns of call options, which are shown in Figures 19 and 20.

Table 5: Model parameters

Panel A: NIG Γ-OU parameters

σ	κ	k	p	q
0.125	0.020	0.800	3.000	3.000

Panel B: Macroeconomic parameters

	a	b	δ
Case 1	3.3000	-1.6612	0.5139
Case 2	3.5000	-1.4805	0.3282
Case 3	3.7000	-1.2853	0.1721
Case 4	3.9000	-1.0737	0.0435

7 Concluding Remarks

This paper has proposed a new dynamic equilibrium model for duplicating some stylized facts observed in stock index markets. In the model, the representative investor has power utility over aggregate consumption and the log-consumption is represented by a linear combination of the log-return of the stock index and its variance. Furthermore, the level of the stock index is driven by a time-changed Lévy process. We have demonstrated that the model is capable of a

Figure 15: Implied Volatility in NIG Γ-OU Model with $T = 1$

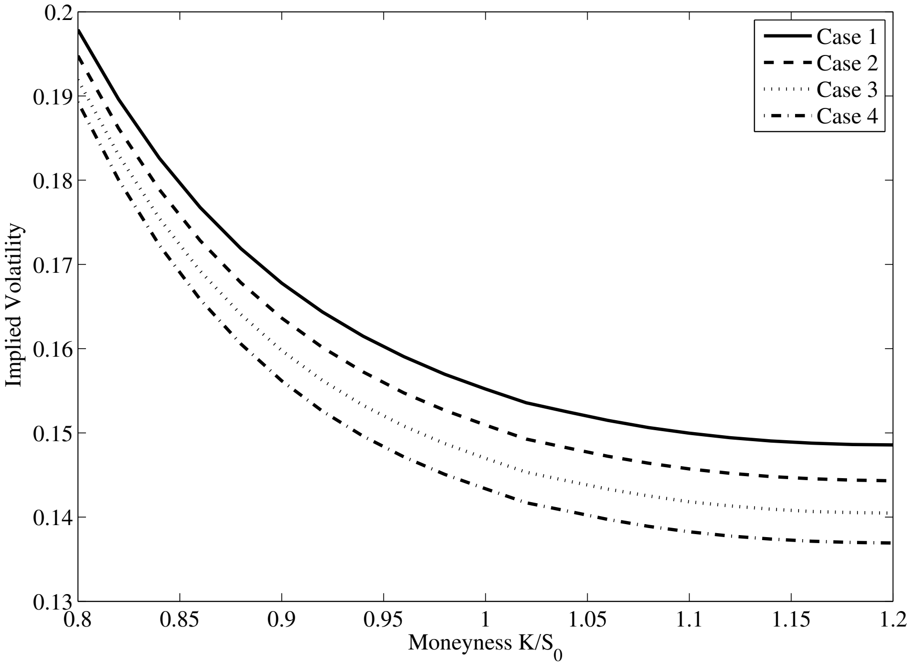


Figure 16: Implied Volatility in NIG Γ-OU Model with $T = 0.5$

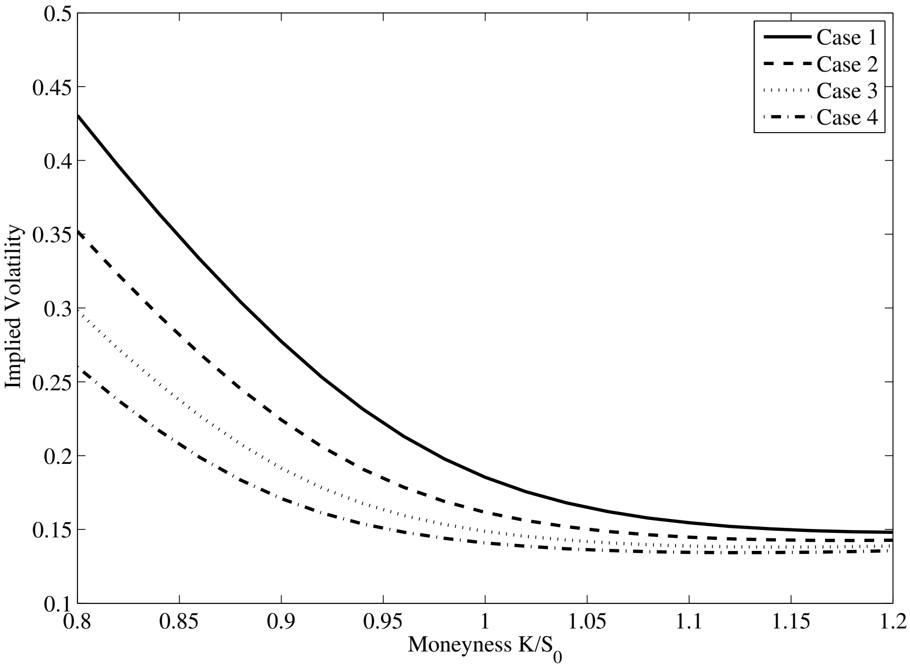


Figure 17: Pricing Kernel in NIG Γ-OU Model with $T = 1$

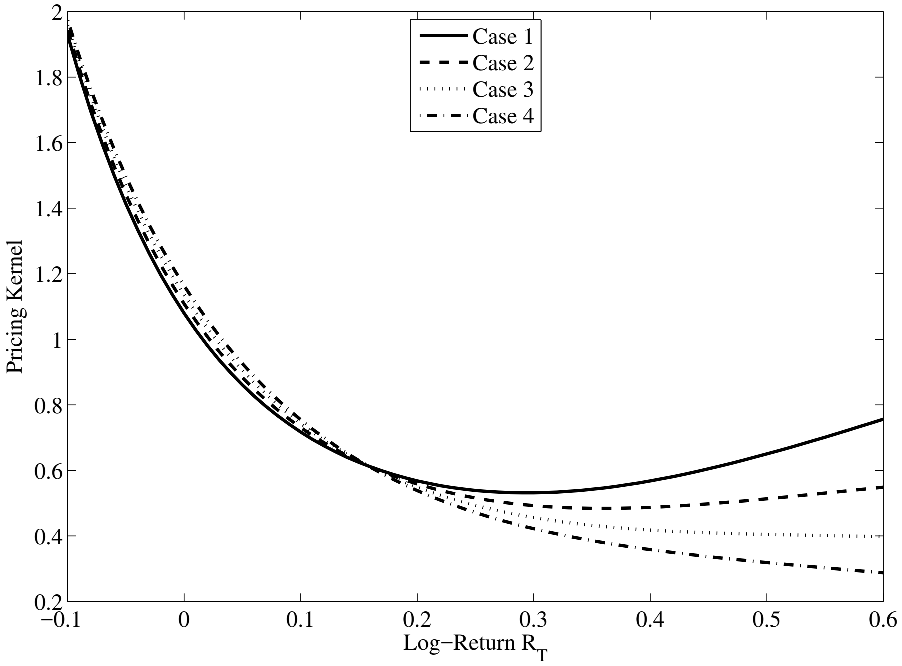


Figure 18: Pricing Kernel in NIG Γ-OU Model with $T = 0.5$

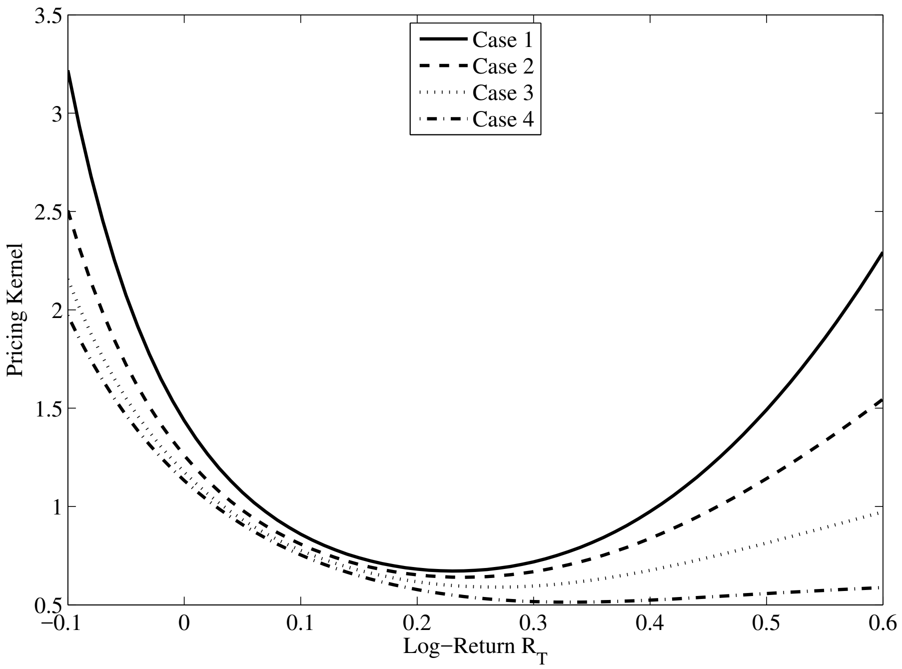


Figure 19: Expected Return of Call Option in NIG Γ-OU Model with $T = 1$

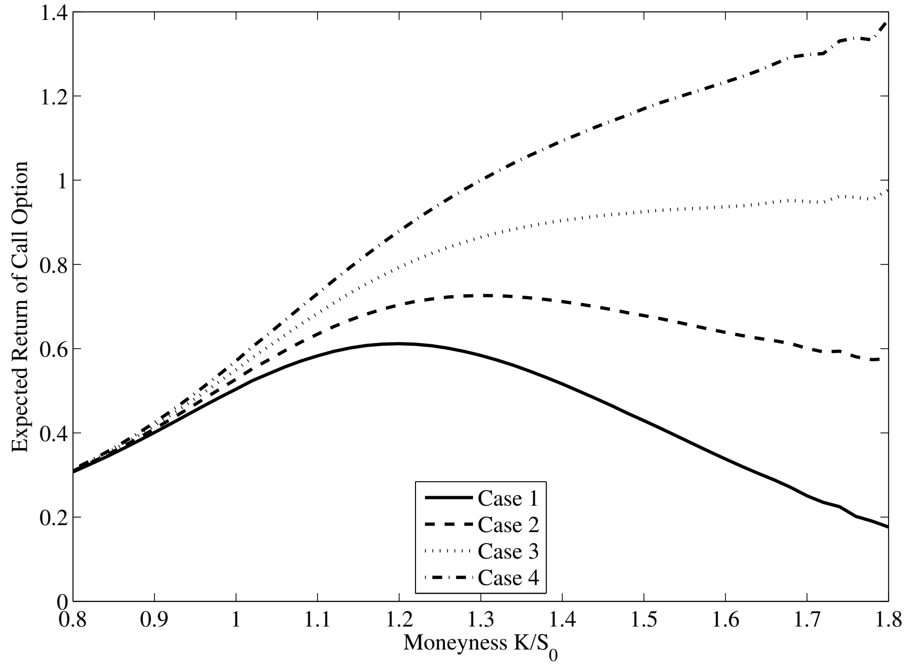


Figure 20: Expected Return of Call Option in NIG Γ-OU Model with $T = 0.5$

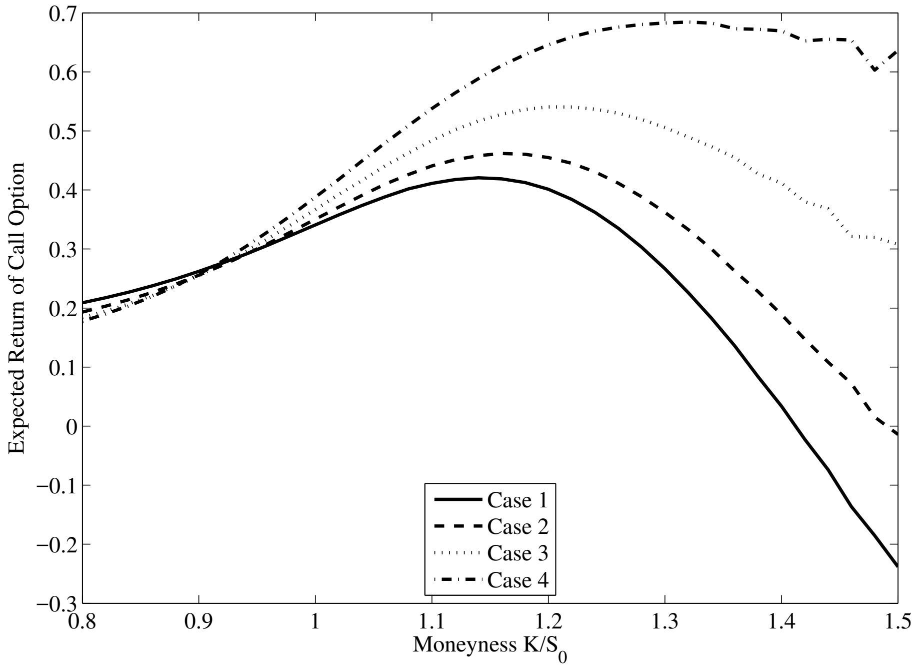


Table 6: Characteristic of distributions and realized variance

Panel A: Fundamental statistics for $T = 1$

	Stdv	RV	Skew	Kurt
$\mathbb{P}$ measure	0.1250	0.1250	-0.0191	0.4007
$\mathbb{Q}$ measure				
Case 1	0.1558	0.1678	-0.3117	0.6890
Case 2	0.1514	0.1589	-0.3110	0.6886
Case 3	0.1473	0.1514	-0.3081	0.6626
Case 4	0.1434	0.1450	-0.3033	0.6500

Panel B: Fundamental statistics for $T = 0.5$

	Stdv	RV	Skew	Kurt
$\mathbb{P}$ measure	0.0884	0.0884	-0.0088	0.3827
$\mathbb{Q}$ measure				
Case 1	0.1061	0.1101	-0.2617	0.7984
Case 2	0.1028	0.1051	-0.2493	0.7696
Case 3	0.0999	0.1011	-0.2362	0.6828
Case 4	0.0974	0.0978	-0.2227	0.6429

unified explanation of the U-shaped pricing kernels, the fat tails of risk-neutral density of the stock index return relative to physical density, and the negative variance risk premium on the stock index. The relationship of the aggregate consumption decreasing in the variance of the stock index return reproduces the U-shaped pricing kernels, which are closely related to the negative expected returns of deep OTM call options on the stock index. We have also shown that even when the stock index return is independent of its variance under the physical measure, the model draws steeply implied volatility skew from the investor's risk aversion. These results lead to a possible theoretical reconciliation of the stylized facts.

The bump shaped term structures of equity risk premiums and interest rates generated by our model are open to discussion. In order to duplicate the positive slope of the term structure of interest rates, one possible alternative is to introduce the recursive utility originated from Epstein & Zin (1989) and Weil (1989), which can be regarded as an extension to power utility. Because recursive utility can separately define the intertemporal elasticity of substitution and risk aversion, one might calibrate the term structures of interest rates and equity risk premiums observed in real markets by incorporating such a utility function.

Finally, we acknowledge that an empirical analysis based on our dynamic equilibrium model remains a matter to be discussed. To do this, we have to develop an estimation method for the model parameters. It goes without saying that a suitable data set is indispensable.

A Some Proofs

A.1 Proof of Proposition 4

The proof we mention here is similar to Carr & Madan (1998) and the chapter 11.2.3 of Cont & Tankov (2004), but slightly different from them.

Define the function h as

$$h(k) := \mathbb{E}^{\mathbb{P}}\left[\left(e^{R_T} - e^k\right)^+\right] - \left(e^{\mu(T)} - e^k\right)^+.$$

Since $\mathbb{E}^{\mathbb{P}}\left[e^{R_T}\right] = e^{\mu(T)}$, we have

$$h(k) = \mathbb{E}^{\mathbb{P}}\left[\left(e^{R_T} - e^k\right)\left(\mathbf{1}_{\{R_T \geq k\}} - \mathbf{1}_{\{\mu(T) \geq k\}}\right)\right].$$

Next, denoting the Fourier transform of h by $g(\theta) := \mathfrak{F}[h](\theta)$, we have

$$\begin{aligned} g(\theta) &= \int_{-\infty}^{\infty} e^{i\theta k} h(k) dk \;=\; \mathbb{E}^{\mathbb{P}}\left[\int_{\mu(T)}^{R_T} e^{i\theta k}\left(e^{R_T} - e^k\right) dk\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\frac{e^{(i\theta+1)R_T}}{i\theta(i\theta+1)} - \frac{e^{i\theta\mu(T)+R_T}}{i\theta} + \frac{e^{(i\theta+1)\mu(T)}}{i\theta+1}\right] \;=\; \frac{\Phi^{\mathbb{P}}_{R_T}(\theta - i) - e^{(i\theta+1)\mu(T)}}{i\theta(i\theta+1)}. \end{aligned}$$

By the inverse Fourier transform of g, the formula (3.21) is obtained. In a similar way, we obtain the formula (3.22) with noting $\mathbb{E}^{\mathbb{Q}}\left[e^{R_T}\right] = e^{r(T)}$. □

A.2 Proof of Proposition 5

Define

$$A_1 := r(t_l) - r(t_{l-1}) - \delta(t_l - t_{l-1}) + i(\theta + ia)\left(\mu(t_l) - \mu(t_{l-1})\right),$$

and

$$A_2 := r(t_{l-1}) - \delta t_{l-1} - a\mu(t_{l-1}).$$

Then, we have

$$\mathbb{E}^{\mathbb{Q}}_{t_{l-1}}\left[e^{i\theta\Delta R_l}\right] = \mathbb{E}^{\mathbb{P}}_{t_{l-1}}\left[\frac{M_{t_l}}{\mathbb{E}^{\mathbb{P}}[M_{t_l}]} \times \frac{\mathbb{E}^{\mathbb{P}}[M_{t_{l-1}}]}{M_{t_{l-1}}} e^{i\theta\Delta R_l}\right] = e^{A_1} I_{l-1},$$

where

$$\begin{aligned} I_{l-1} &:= \mathbb{E}^{\mathbb{P}}_{t_{l-1}}\left[e^{(i\theta-a)\left(X_{t_l} - X_{t_{l-1}}\right) - (i\theta-a)\varphi_Y(-i)\left(\tau_{t_l} - \tau_{t_{l-1}}\right) - b\left(v_{t_l} - v_{t_{l-1}}\right)}\right] \\ &= \mathbb{E}^{\mathbb{P}}_{t_{l-1}}\left[\frac{Z_{t_l}(\theta+ia)}{Z_{t_{l-1}}(\theta+ia)} \exp\left\{\lambda(\theta+ia)\int_{t_{l-1}}^{t_l} v_{s-} ds - b\left(v_{t_l} - v_{t_{l-1}}\right)\right\}\right] \\ &= \mathbb{E}^{\mathbb{P}(\theta+ia)}_{t_{l-1}}\left[\exp\left\{\lambda(\theta+ia)\int_{t_{l-1}}^{t_l} v_{s-} ds - b\left(v_{t_l} - v_{t_{l-1}}\right)\right\}\right]. \end{aligned}$$

Using the fact that $A_1 + A_2 = -b - \psi_{t_l}(-a, -b) + i\theta(\mu(t_l) - \mu(t_{l-1}))$ from Proposition 2, we have

$$\begin{aligned} \Phi^{\mathbb{Q}}_{\Delta R_l}(\theta) &= \mathbb{E}^{\mathbb{Q}}\left[e^{i\theta\Delta R_l}\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}_{t_{l-1}}\left[e^{i\theta\Delta R_l}\right]\right] = e^{A_1}\mathbb{E}^{\mathbb{P}}\left[\frac{M_{t_{l-1}}}{\mathbb{E}^{\mathbb{P}}[M_{t_{l-1}}]} I_{l-1}\right] \\ &= e^{A_1 + A_2}\mathbb{E}^{\mathbb{P}}\left[Z_{t_{l-1}}(ia)\exp\left\{\lambda(ia)\tau_{t_{l-1}} - b(v_{t_{l-1}} - 1)\right\} I_{l-1}\right] \\ &= \frac{e^{i\theta(\mu(t_l) - \mu(t_{l-1}))}}{\Psi^0_{t_l}}\mathbb{E}^{\mathbb{P}(ia)}\left[\exp\left\{\lambda(ia)\tau_{t_{l-1}} - b v_{t_{l-1}}\right\} I_{l-1}\right]. \end{aligned}$$

Note that $\frac{d\mathbb{Q}}{d\mathbb{P}}|_t = 1$ and $\mu(t) = 0$ for any $t \geq 0$ when $\delta = \gamma = 0$. Therefore, putting $\delta = \gamma = 0$ in (3.28), we obtain (3.27). □

B Moments of Distribution of Realized Variance

Applying the multinomial formula, we represent the nth moment of the risk-neutral distribution of the realized variance as

$$\mathbb{E}^{\mathbb{Q}}\left[V_T^n\right] = \sum_{k_1+k_2+\cdots+k_L=n} \frac{n!}{k_1!k_2!\cdots k_L!}\mathbb{E}^{\mathbb{Q}}\left[\prod_{l=1}^{L}(\Delta R_l)^{2k_l}\right]. \tag{B.1}$$

The nth moment of the physical distribution of the realized variance is represented in a similar fashion. Following to (B.1), we only need to have any cross-moments of increments of the log-returns to obtain an arbitrary moment of the realized variance. Therefore, we will derive useful expressions of the characteristic functions of the physical and the risk-neutral distributions of the $\mathbf{R}^L$-valued random variable $\Delta R := (\Delta R_1, \cdots, \Delta R_L)^\top$.

Proposition 6 *(Multivariate Characteristic Function under Risk-Neutral Measure) Define the characteristic function of the risk-neutral distribution of ΔR as*

$$\Phi^{\mathbb{Q}}_{\Delta R}(\Theta) := \mathbb{E}^{\mathbb{Q}}\left[e^{i\Theta^\top \Delta R}\right], \tag{B.2}$$

where $\Theta = (\theta_1, \ldots, \theta_L)^\top \in \mathcal{D}^L$. Let $(J_l^{\mathbb{Q}})_{0\le l\le L}$ be a backward recurrence relation such that

$$J^{\mathbb{Q}}_{l-1} = \mathbb{E}^{\mathbb{P}(\theta_l+ia)}_{t_{l-1}}\left[\exp\left\{\lambda(\theta_l+ia)\int_{t_{l-1}}^{t_l} v_{s-}ds - b(v_{t_l}-v_{t_{l-1}})\right\}J^{\mathbb{Q}}_l\right], \tag{B.3}$$

for $l = 1, \ldots, L$, where the terminal condition is $J^{\mathbb{Q}}_L = e^{-b}$ and $\lambda(x)$ is the function defined in (3.3). Then, the characteristic function is represented as

$$\Phi^{\mathbb{Q}}_{\Delta R}(\Theta) = \exp\left\{i\sum_{l=1}^{L}\theta_l(\mu(t_l)-\mu(t_{l-1}))\right\}\frac{J_0^{\mathbb{Q}}}{\Psi_T^0}. \tag{B.4}$$

Proof of Proposition 6: Define the backward recurrence relation

$$K_{l-1} = \mathbb{E}^{\mathbb{Q}}_{t_{l-1}}\left[e^{i\theta_l\Delta R_l}K_l\right], \quad \text{for } l = 1, \ldots, L,$$

with $K_L = 1$, and the sequence $(B_l)_{1\le l\le L}$

$$B_l = r(t_l) - r(t_{l-1}) + i(\theta_l + ia)(\mu(t_l) - \mu(t_{l-1})) - \delta(t_l - t_{l-1}).$$

Next, we have

$$\begin{aligned} K_{l-1} &= \mathbb{E}^{\mathbb{P}}_{t_{l-1}}\left[\frac{M_{t_l}}{\mathbb{E}^{\mathbb{P}}[M_{t_l}]}\times\frac{\mathbb{E}^{\mathbb{P}}[M_{t_{l-1}}]}{M_{t_{l-1}}}e^{i\theta_l\Delta R_l}K_l\right] \\ &= e^{B_l}\mathbb{E}^{\mathbb{P}}_{t_{l-1}}\left[e^{(i\theta_l-a)\left(X_{t_l}-X_{t_{l-1}}\right)-(i\theta_l-a)\varphi_Y(-i)\left(\tau_{t_l}-\tau_{t_{l-1}}\right)-b\left(v_{t_l}-v_{t_{l-1}}\right)}K_l\right] \\ &= \exp\left\{b+\sum_{m=l}^{L}B_m\right\}J^{\mathbb{Q}}_{l-1}, \end{aligned}$$

where we define

$$J^{\mathbb{Q}}_{l-1} := \mathbb{E}^{\mathbb{P}}_{t_{l-1}}\left[e^{(i\theta_l-a)\left(X_{t_l}-X_{t_{l-1}}\right)-(i\theta_l-a)\varphi_Y(-i)\left(\tau_{t_l}-\tau_{t_{l-1}}\right)-b\left(v_{t_l}-v_{t_{l-1}}\right)}J^{\mathbb{Q}}_l\right],$$

with $J_L^{\mathbb{Q}} = e^{-b}$. Then, it is shown that

$$\begin{aligned} J_{l-1}^{\mathbb{Q}} &= \mathbb{E}_{t_{l-1}}^{\mathbb{P}}\left[\frac{Z_{t_l}(\theta_l + ia)}{Z_{t_{l-1}}(\theta_l + ia)} \exp\left\{\lambda(\theta_l + ia)\int_{t_{l-1}}^{t_l} v_{s-}ds - b(v_{t_l} - v_{t_{l-1}})\right\} J_l^{\mathbb{Q}}\right] \\ &= \mathbb{E}_{t_{l-1}}^{\mathbb{P}(\theta_l + ia)}\left[\exp\left\{\lambda(\theta_l + ia)\int_{t_{l-1}}^{t_l} v_{s-}ds - b(v_{t_l} - v_{t_{l-1}})\right\} J_l^{\mathbb{Q}}\right]. \end{aligned}$$

Noting that

$$b + \sum_{l=1}^{L} B_l = -\psi_T(-a, -b) + i\sum_{l=1}^{L}\theta_l(\mu(t_l) - \mu(t_{l-1})),$$

and $K_0 = \Phi_{\Delta R}^{\mathbb{Q}}(\Theta)$ by the law of iterated expectations, the proof is completed. □

Assigning zeros to δ and γ in (B.3) and (B.4), the following corollary is obtained.

Corollary 2 *(Multivariate Characteristic Function under Physical Measure) Define the characteristic function of the physical distribution of ΔR as*

$$\Phi_{\Delta R}^{\mathbb{P}}(\Theta) := \mathbb{E}^{\mathbb{P}}\left[e^{i\Theta^\top \Delta R}\right], \tag{B.5}$$

and let $(J_l^{\mathbb{P}})_{0\le l\le L}$ be a backward recurrence relation such that

$$J_{l-1}^{\mathbb{P}} = \mathbb{E}_{t_{l-1}}^{\mathbb{P}(\theta_l)}\left[\exp\left\{\lambda(\theta_l)\int_{t_{l-1}}^{t_l} v_{s-}ds\right\} J_l^{\mathbb{P}}\right],$$

where the terminal condition is $J_L^{\mathbb{P}} = 1$. Then, the characteristic function is represented as

$$\Phi_{\Delta R}^{\mathbb{P}}(\Theta) = J_0^{\mathbb{P}}.$$

The closed-form expressions of the multivariate characteristic functions of (B.2) and (B.5) are obtained if the characteristic function of the bivariate random variable (τ_t, v_t) has a closed-form expression. However, even in such a case, calculating a cross moment of increments of the log returns, equivalently, a moment of the realized variance, is not easy task because obtaining the values of the cross moments in (B.1), which are written as

$$\mathbb{E}^{\mathbb{Q}}\left[\prod_{l=1}^{L}(\Delta R_l)^{2k_l}\right] = \frac{\partial^{2n}}{\partial^{2k_1}\cdots\partial^{2k_1}}\Phi_{\Delta R}^{\mathbb{Q}}(\Theta)\Big|_{\Theta=\mathbf{0}},$$

has to do tedious calculations in general.

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