WORKING PAPER SERIES

Akira Yamazaki

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January 29, 2019

No. 201

The Research Institute for Innovation Management, HOSEI UNIVERSITY

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Akira Yamazaki
Graduate School of Business Administration, Hosei University
The Research Institute for Innovation Management, Hosei University
2-17-1, Fujimi, Chiyoda-ku, TOKYO 102-8160, JAPAN
Phone: +81-3-3264-6450, E-mail: akira2yamazaki@hosei.ac.jp

January 27, 2019

Abstract

This paper provides an equation called the *pricing kernel equation*, which relates the subjective probability distribution on an arbitrary asset price to the risk-neutral probability distribution. It claims that the subjective probability distribution is priced by a static option portfolio, in which the weight of an option is determined by the level of the pricing kernel. As an application, we propose a new method for estimating empirical pricing kernels. Another application is to extract subjective probabilities and fundamental statistics from option prices. These examples show that the pricing kernel equation can be a versatile tool for various applications.

Keywords: subjective probability; risk-neutral probability; pricing kernel equation; reciprocal kernel; option market; static replication; absolute risk aversion

*Akira Yamazaki, Graduate School of Business Administration, Hosei University. The author gratefully acknowledges the financial support from JSPS KAKENHI, Grant Number 26380402. His work is also supported by the Research Institute for Innovation Management at Hosei University.

1 Introduction

Due to the progress of modern finance theory, the principle of asset pricing has been established in a rigorous manner. The price of an asset in equilibrium is evaluated as the expected value of its payout multiplied by a pricing kernel. More precisely, the expected value is taken not under an objective probability measure, but under a subjective probability measure associated with a market consensus. Although the pricing kernel also known as the stochastic discount factor or the state-price density plays a central role in asset pricing, it is not so easy to specify its explicit form adequately. Another asset pricing formula states that the asset price equals the risk-neutral expected value of its payout discounted by a risk-free interest rate. As a matter of course, the former is equivalent to the later in theory.

The objective of this paper is to have a rethink about the relation among three building blocks; the subjective probability distribution, the risk-neutral probability distribution, and the pricing kernel. The relation has already been well recognized as an equation that the pricing kernel equals the likelihood ratio of the risk-neutral probability to the subjective probability discounted by a risk-free interest rate. It is important for us to become conscious of the fact that the likelihood ratio is usually written as a form in terms of *probability density functions*. One question of this paper is how to represent the relation in terms of *cumulative distribution functions* as a general formula. The derived representation called the *pricing kernel equation* is expected to be more fruitful than the already known representation of the pricing kernel. Another question is how to describe subjective fundamental statistics, including mean, standard deviation, skewness, and kurtosis of the subjective probability distribution, by being estimated from observed market data and a given pricing kernel. We also explore a new general formulation of the pricing kernel. Moreover, we suggest some applications of the pricing kernel equation.

For this purpose, we begin with the projection of the pricing kernel onto the price of a target asset and then consider its reciprocal, which is called the *reciprocal kernel*. It is a convenient expression to relate the subjective probability distribution on the asset price to the risk-neutral one. Applying the reciprocal kernel to a static replication strategy, we derive the pricing kernel equation that is a fundamental formula for any pricing kernels. This formula is represented as an integro-differential equation satisfying the reciprocal kernel. The pricing kernel equation describes the explicit linkage between the subjective and the risk-neutral cumulative distribution functions. It claims that the subjective cumulative distribution function is priced by a static option portfolio consisted of a digital put option and a series of plain vanilla put options written on the asset. The weight of a put option in the static portfolio is determined by the level of the reciprocal kernel or its differentiation. It is also proved that a general solution to the pricing kernel equation exits under some conditions.

Our approach to derive the pricing kernel equation is a static replication argument by an option portfolio to quote the subjective probability distribution as a price. Static replication strategies composed of available options in markets have been an accepted approach in both academia and practice. For example, Takahashi & Yamazaki (2009a, 2009b), Ohsaki & Yamazaki (2011), and Carr & Wu (2014) replicated long-term options or defaultable bonds by a static portfolio of plain vanilla options. It is well known that variance swaps or their subspecies can be replicated by a static option portfolio combined with dynamic trading of an underlying asset. For specific information, see Carr & Madan (1998), Demeterfi et al. (1999), Carr & Lewis (2004), Schoutens (2005), and Takahashi et al. (2011), among others. Bakshi et al. (2003) and Martin (2017) exploited ingenious estimation methods for fundamental statistics of asset prices by static replication strategies. By contrast, this paper applies such a static replication strategy to derive the fundamental principle of asset pricing. However, we acknowledge that the conception of the pricing kernel equation is considerably inspired by these previous studies.

As an application of the pricing kernel equation, we suggest a new estimation method of empirical pricing kernels. As regards theoretical research, Lucas (1978) documented the concept of the pricing kernel as a marginal utility form, which is a monotonically decreasing function of aggregate consumption. Thereafter, it has been assumed to have such a form of the pricing kernel in the standard theory of asset pricing. In contrast, Aït-Sahalia & Lo (2000), Jackwerth (2000), and Rosenberg & Engle (2002), among others discovered that empirical pricing kernels are not monotonically decreasing in the underlying state variable, but have some increasing region. More precisely, the empirical pricing kernels depict U-shaped or tilde-shaped curves. This inconsistency is known as the *pricing kernel puzzle*. Recent empirical analysis, including Fengler & Hin (2015) and Song & Xiu (2016), reaches the same conclusion. To resolve the pricing kernel puzzle, Bakshi et al. (2010), Christoffersen et al. (2013), and Yamazaki (2018) exploited asset pricing models equipped with U-shaped pricing kernels. Cuesdeanu & Jackwerth (2018) demonstrated that the ambiguity aversion model of Klibanoff et al. (2005) naturally generates a tilde-shaped pricing kernel. In previous empirical research, an empirical distribution based on historical returns of a stock index has been identified with the subjective probability distribution. This paper provides a simulation test for empirical pricing kernel estimation in which a plausible projected pricing kernel function is given beforehand. Given a series of stock index returns generated by a Monte Carlo simulation and an implied volatility curve, we first estimate both empirical and risk-neutral cumulative distribution functions. And then plugging the two distribution functions into the pricing kernel equation, we estimate the shape of the pricing kernel function. The simulation test verifies the advantages of the estimation method based on the pricing kernel equation in comparison with a standard estimation method. As a by-product, it is demonstrated that empirical pricing kernels are essentially uncertain as long as a data set available in practice are used for. Notably, empirical results reporting that oscillating pricing kernels are observed are dubious.

Another application is to extract the subjective probability distribution. Tversky & Kahneman (1992) focused on the discrimination between subjective and objective probabilities in the cumulative prospect theory, which is a modified version of the prospective theory of Kahneman & Tversky (1979). In the theory, the probability weighting function characterizes the distortion of subjective probabilities to objective ones. A large number of experimental research, including Tversky & Kahneman (1992), Wu & Gonzalez (1996), Prelec (1998), Berns et al. (2007), and Polkovnichenko & Zhao (2013), supported the existence of the probability weighting function and reported that subjective probabilities are definitely different from objective ones. Taking the empirical results, Barberis & Huang (2008) and Yamazaki (2019) incorporated the probability weighting function into asset pricing models to explain negative average returns on IPO stocks and financially distressed stocks, respectively. This paper provides a numerical example for extracting the subjective probability distribution from observed option prices. We first develop the functional form of the reciprocal kernel consistent with the hyperbolic absolute risk aversion utility. Subsequently, we try to estimate subjective probabilities and subjective fundamental statistics from an implied volatility curve given beforehand. In general, in order to obtain the pricing kernel associated with a utility function, one has to know subjective probabilities in advance and then to solve a consumption-investment problem under the subjective probabilities. By contrast, we do not have to do so in this application. What we have to do is to construct a simple static option portfolio in order to determine the reciprocal kernel and to input it into the pricing kernel equation. This is the reason why the subjective probability distribution can be restored from observed option prices and a given utility function.

The remainder of this paper proceeds as follows. Section 2 describes a preliminary example to make our objectives clear. Section 3 provides the pricing kernel equation and solves it. Section 4 shows the simulation test for verifying empirical pricing kernels. In Section 5, we discuss the reciprocal kernel and its applications. Section 6 presents the numerical example that estimates subjective probability distributions. Section 7 concludes. Further, the Appendices contain some technical supplements and detailed discussion.

2 A Preliminary Example

First of all, consider a simple one-period consumption-investment problem with a representative investor as an example to motivate the following sections. Let W_t and C_t be his/her wealth and consumption at date $t \in \{0, T\}$, respectively. Suppose that the investor can choose to invest in M different risky assets. Let $S_{m,0}$ be the price per share of asset $m \in \{1, \ldots, M\}$ at date 0 and $S_{m,T}$ be the random payoff of asset m at date T. Define the gross return on asset m as $R_m := S_{m,T}/S_{m,0}$. It is assumed that the investor does not receive any labor income for simplicity. Thus, his/her budget constraint is

$$C_T = (W_0 - C_0) \sum_{m=1}^{M} w_m R_m, \tag{2.1}$$

where w_m denotes the proportion of investment in asset m at date 0 such that $\sum_m w_m = 1$. Recall that the net supply of the risk-free asset is zero in economy with a representative investor. The consumption-investment problem can then be stated as

$$\max_{C_0, \{w_m\}} U(C_0) + \delta \mathbb{E}\left[U(C_T)\right], \tag{2.2}$$

where $\delta \in (0, 1)$ is some constant denoting a time preference discount factor and $U(\,\cdot\,)$ is the investor's utility function. Here, we would like to stress that $\mathbb{E}[\,\cdot\,]$ denotes the expectation operator under the *investor's subjective probability* $\mathbb{P}$. Solving consumption-investment problem (2.2), the asset pricing formula is obtained as the form

$$S_{m,0} = \mathbb{E}\left[\delta \frac{U'(C_T^{op})}{U'(C_0^{op})} S_{m,T}\right], \quad \text{for} \quad m = 1, \ldots, M, \tag{2.3}$$

where C_t^{op} denotes the optimal consumption at date t. In equilibrium, the return on the optimal portfolio of all risky assets becomes the return on the market portfolio and its terminal value is equal to the optimal consumption C_T^{op} realized by the optimal investment weights $\{w_m^{op}\}$. In Eq.(2.3),

$$\delta \frac{U'(C_T^{op})}{U'(C_0^{op})}, \tag{2.4}$$

is so-called the pricing kernel. That is, the current price of an arbitrary asset equals the expected value of its payoff multiplied by the pricing kernel. It is thought that the pricing kernel is an essential component for all of modern finance.

According to budget constraint (2.1), pricing kernel (2.4) can be regarded as a function of the gross return on the market portfolio or an asset. For that matter, it is a strictly positive and decreasing function of the terminal payoff of the market portfolio or an asset if the utility function is strictly increasing and concave. This is the reason why a number of researchers have estimated the shapes of empirical pricing kernels projected onto the values of a stock index. In such empirical studies, the investor's subjective probability distribution has been identified with an empirical probability distribution based on historical return data, but probability measure $\mathbb{P}$ in Eqs.(2.2) and (2.3) is obviously not so. The objective of this paper is just the *subjective probability distribution.*

3 Pricing Kernel Equation

Let $\mathcal{M}$ be a strictly positive random variable such that the current price of an arbitrary payout X paid at time T is written as

$$\mathbb{E}\left[\mathcal{M}X\right].$$

That is to say, $\mathcal{M}$ is the pricing kernel. Next, define the projection of the pricing kernel onto the time-T price of an asset as

$$\mathcal{M}(S_T) := \mathbb{E}\left[\mathcal{M} \,|\, S_T\right]. \tag{3.1}$$

The object of an asset for projection (3.1) is usually considered as the market portfolio in theory, or a stock index in practice, but it can be chosen arbitrarily in our framework. Hereafter, the projected pricing kernel $\mathcal{M}(x)$ is assumed to be twice differentiable with respect to $x > 0$. Recall that the existence of a strictly positive pricing kernel is a necessary and sufficient condition that there are no arbitrage opportunities. The twice differentiability of the projected pricing kernel means that the representative investor has a three-times differentiable utility function over the asset price in the context of the preliminary example in Section 2.

Throughout this paper, we frequently use the reciprocal of the projected pricing kernel

$$g(x) := 1/\mathcal{M}(x),$$

which is named the *reciprocal kernel*. Of course, inquiry into the reciprocal kernel is equivalent to that into the projected pricing kernel. However, as will be shown below, it is convenient for us to treat the reciprocal kernel directly. With the assumptions on the pricing kernel mentioned above, the reciprocal kernel is also strictly positive and twice differentiable. The next lemma is trivial, but essential for characterizing the subjective probability distribution.

Lemma 1 *The subjective expected value of an arbitrary payoff depending on the terminal asset price S_T denoted by $H(S_T)$ equals the price of the product of the reciprocal kernel $g(S_T)$ and $H(S_T)$. That is,*

$$\mathbb{E}\left[H(S_T)\right] = \frac{1}{R_f}\mathbb{E}_*\left[g(S_T)H(S_T)\right], \tag{3.2}$$

where $\mathbb{E}_*[\,\cdot\,]$ denotes the expectation operator under the risk-neutral (forward-neutral) probability measure $\mathbb{Q}$ and R_f is the gross risk-free rate.

Proof of Lemma 1: By equivalence of the asset pricing formulas, the following equality holds for an arbitrary payoff X paid at time T.

$$\mathbb{E}\left[\mathcal{M}X\right] = \frac{1}{R_f}\mathbb{E}_*\left[X\right]. \tag{3.3}$$

Substituting $X = g(S_T)H(S_T)$ into Eq.(3.3) and applying the law of iterated expectations to the left side of Eq.(3.3) leads to Eq.(3.2). □

The next statement, which is the main theorem of this paper, presents a model-free equation representing the explicit relation between the subjective and the risk-neutral probability distributions on S_T mediated by the reciprocal kernel and its derivatives.

Theorem 1 (Pricing Kernel Equation) *The subjective cumulative distribution function of the asset price at time T denoted by $F(x) := \mathbb{P}(S_T \le x)$ is represented as*

$$F(x) = \frac{1}{R_f} g(x) F_*(x) - g'(x) P(T,x) + \int_0^x g''(K) P(T,K) dK, \tag{3.4}$$

where $F_*(x) := \mathbb{Q}(S_T \le x)$ denotes the risk-neutral cumulative distribution function of the asset price at time T and $P(T,K)$ denotes the current price of a put option written on the asset maturing at time T with strike price K.

Proof of Theorem 1: According to Carr & Madan (1998), $g(S_T)$ can be expressed as

$$g(S_T) = g(x) + g'(x)(S_T - x) + \int_0^x g''(K)(K - S_T)^+ dK + \int_x^\infty g''(K)(S_T - K)^+ dK, \tag{3.5}$$

for any $x > 0$. Multiplying the both side of Eq.(3.5) by $\mathbf{1}_{\{S_T \le x\}}$ yields

$$\mathbf{1}_{\{S_T \le x\}} g(S_T) = \mathbf{1}_{\{S_T \le x\}} g(x) - g'(x)(x - S_T)^+ + \int_0^x g''(K)(K - S_T)^+ dK. \tag{3.6}$$

Right here, taking $H(S_T) = \mathbf{1}_{\{S_T \le x\}}$ in Lemmma 1 and substituting the right side of Eq.(3.6) into the right side of Eq.(3.3), we obtain Eq.(3.4). $\square$

Theorem 1 has three interpretations: First, it proves the explicit linkage between the subjective and the risk-neutral cumulative distribution functions. That is, Eq.(3.4) is a model-free equation to relate the subjective cumulative distribution function to the risk-neutral one and it is composed of the market prices of put options and an arbitrarily given reciprocal kernel. It is worthwhile noting that Eq.(3.4) is not about the probability density functions, but about the cumulative distribution functions. To the best of our knowledge, Eq.(3.4) is the first equation to give a general formulation of the relation between the subjective and the risk-neutral cumulative distribution functions. This fact would be a strong point when applying Theorem 1. Consider a simple consistency check. If the representative investor in the preliminary example of Section 2 is risk-neutral, then $g(x) = 1/\delta$. As a result, his/her subjective cumulative distribution function is coincident with the risk-neutral one due to $R_f = 1/\delta$ and $g'(x) = g''(x) = 0$.

Second, Theorem 1 can be interpreted that the subjective cumulative distribution function is priced by a *static option portfolio*. It is constituted of a long position in $g(x)$ units of cash digital put options struck at x, whose unit price is $F_*(x)/R_f$, a short position in $g'(x)$ units of plain vanilla put options struck at x, and a long position in $g''(K)dK$ units of plain vanilla put options at all strikes less than x. This portfolio is static, which means that an investor invests in these positions at initial time and holds them until maturity. In other words, the subjective probability distribution on the time-T price of an asset is marketed by options written on it.

Third, Eq.(3.4) can be regarded as the integro-differential equation of the reciprocal kernel, $g(x)$. Obviously, it is also the integro-differential equation of the pricing kernel itself, $\mathcal{M}(x)$. Therefore, we call it the *pricing kernel equation*, or the PKE for short. In general, it is difficult to solve an integro-differential equation explicitly and it may not have a closed-form solution. In such cases, it is solved numerically by a finite-difference method for instance. Fortunately, the PKE has the explicit solution under some conditions and it can be easily obtained. The form of the solution must be familiar with financial economists. The following theorem presents the solution to PKE (3.4).

Theorem 2 (Solution) *If both of the subjective and the risk-neutral cumulative distribution functions are absolutely continuous, the PKE has the solution*

$$g(x) = R_f \frac{\phi(x)}{\phi_*(x)}, \tag{3.7}$$

where $\phi(x)$ and $\phi_*(x)$ denote the subjective and the risk-neutral probability density functions of the asset price at time T, respectively.

Proof of Theorem 2: Differentiating the both sides of PKE (3.4) with respect to x and substituting

$$\frac{d}{dx} P(T, x) = \frac{1}{R_f} F_*(x),$$

into it, we obtain solution (3.7). □

The reciprocal of the right side of Eq.(3.7) is a standard representation of the pricing kernel in past literature. However, the condition on Theorem 2 that the subjective and the risk-neutral probability distributions have the density functions is restrictive. Accordingly, we avoid the use of the density functions $\phi(x)$ and $\phi_*(x)$ as long as possible.

4 Simulation Test

An application of the PKE is to estimate empirical pricing kernels from observed market data. In past empirical studies, an empirical probability distribution based on historical returns of a stock index as a proxy for the market portfolio has been identified with the associated subjective probability distribution. Therefore, an empirical pricing kernel is computed by the reciprocal of Eq.(3.7) that is the discounted value of the ratio of a risk-neutral density function implied from observed option prices to an empirical density function. This approach originally attempted by Jackwerth (2000) is thought of a standard method to estimate the shape of empirical pricing kernels. Similarly, making use of an empirical cumulative distribution function and implied risk-neutral cumulative distribution function, we can obtain an empirical pricing kernel by numerically solving the PKE. This approach is named the *pricing kernel equation estimation* or the PKE estimation for short. In comparison with the standard method, conceivable advantages of the PKE estimation are as follows: First, the PKE estimation needs not assume the absolutely continuity of the subjective and the risk-neutral cumulative distribution functions. In general, it is difficult to verify the absolutely continuity. Because of this, a number of researchers using the standard method have implicitly assumed the absolutely continuity as a premise of their analysis. Second, the PKE estimation does not use an empirical density function, but an empirical cumulative distribution function based on historical returns or values of a stock index. On the other hand, the standard method needs an empirical probability density function. When applying the kernel density estimation method that is a non-parametric way to estimate a probability distribution from empirical data, the convergence rate of an empirical cumulative distribution function to the exact distribution is faster than that of an empirical density function. More detailed discussion about the kernel density estimation can be found in Appendix B. In practice, only a limited number of historical values of a stock index has to be used for such estimation methods. For example, Jackwerth (2000) used only 48 non-overlapping monthly returns on the S&P 500. Therefore, faster convergence speed is crucial. Third, the PKE estimation uses an implied risk-neutral cumulative distribution function based on observed option prices. To obtain it, we need *first-order* differentiation of an implied volatility function with respect to strike prices. In the standard method, *second-order* differentiation of an implied volatility function is needed to obtain an implied risk-neutral probability density function. More details are discussed in Appendix C. In finite-difference methods, numerical errors of second-order differentiation are larger than that of first-order differentiation in general. This fact is commonly known as the *curse of differentiation.* Aït-Sahalia & Lo (1998) illustrated such numerical errors in their figure 2, which depicts that the errors of the implied risk-neutral density are larger than those of the implied risk-neutral cumulative distribution. Moreover, in the standard method, one has to divide a vulnerable empirical density function by a vulnerable implied risk-neutral density function in order to obtain an empirical pricing kernel. In particular, the division of very small unstable values on the tails of the two density functions is problematic.

In the following, we attempt a simulation test, in which we estimate a given pricing kernel projected on a stock index by the PKE estimation and the standard method, respectively. And then, we verify each estimation ability. In the simulation test, the subjective probability distribution is identified with an empirical one in accordance with past empirical studies. Because of this, the estimated pricing kernel is here called the *empirical pricing kernel.* Suppose that S_t is the time-t value of a stock index as a proxy for the market portfolio and generated by Heston's stochastic volatility model (Heston, 1993). That is, under the subjective probability measure $\mathbb{P}$, the values of the stock index are governed by

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma\sqrt{v_t}dB_t^1, \quad \text{with} \quad S_0 = 1, \\ dv_t &= k(1-v_t)dt + c\sqrt{v_t}\left(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2\right), \quad \text{with} \quad v_0 = 1, \end{aligned} \tag{4.1}$$

where B^1 and B^2 are independent standard Brownian motions under $\mathbb{P}$, and $\mu, \sigma, k, c > 0$ and $\rho \in [-1, 1]$ are some constants with the parameter restriction $2k > c^2$, which is known as the Feller condition ensuring that the stochastic process v remains strictly positive at any time. For simplicity, the initial values of S and v are normalized to be ones. The values of the Heston model parameters are exhibited in Panel A of Table 1.

The first step is that, by Monte Carlo simulation of Eq.(4.1), we generate a series of annual gross returns on the stock index postulated as observed sample data. Notice that the annual gross returns equal the associated values of the stock index over the next year in the simulation test owing to the normalization. As the second step, we assign the risk-neutral probability $\mathbb{Q}$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{S_T} := \frac{S_T^{2\alpha} + \beta S_T^{\alpha} + \gamma}{\mathbb{E}\left[S_T^{2\alpha} + \beta S_T^{\alpha} + \gamma\right]}, \tag{4.2}$$

with some constants α, β, and γ. The values of these parameter are listed in Panel B of Table 1. The exact projected pricing kernel that is the target for our estimation is written as

$$\mathcal{M}(x) = \frac{1}{R_f}\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{x} = e^{-r(T)}\frac{x^{2\alpha} + \beta x^{\alpha} + \gamma}{\mathbb{E}\left[S_T^{2\alpha} + \beta S_T^{\alpha} + \gamma\right]}, \tag{4.3}$$

where $r(T) := \log R_f$ is the one-year yield to maturity in market equilibrium. The closed form expressions of $\mathcal{M}(x)$ and $r(T)$ can be found in Appendix A. Note that the pricing kernel defined in Eq.(4.3) depicts U-shaped curve because it is a quadratic function of x^{α}. This is consistent with some empirical observations. For example, Jackwerth (2000), Bakshi et al. (2010), and Christoffersen et al. (2013) reported that U-shaped empirical pricing kernels have been observed in stock index markets. However, this is inconsistent with the marginal utility form in Eq.(2.4) of the pricing kernel, which is monotonically decreasing in x. Next, we compute prices of plain vanilla call options written on the stock index by the option pricing formula given in Appendix

A.2. The option maturity is one-year and strike price ranges from 0.8 to 1.4. As shown in Figure 1, we then compute the Black-Scholes implied volatilities from the call option prices obtained above.

In the simulation test, we presume that observers only know sample values of the stock index, the one-year yield to maturity, and implied volatilities of the stock index. They estimate pricing kernel (4.3) by the PKE estimation or the standard method. In the PKE estimation, we use kernel density estimation (B.3) in Appendix B for empirical cumulative distribution functions and formula (C.1) in Appendix C for risk-neutral one. Then, we apply the point-wisely quadratic approximation described in Appendix D to solve the PKE numerically. In the standard method, we use kernel density estimation (B.1) in Appendix B for empirical density functions and formula (C.2) in Appendix C for risk-neutral one.

Table 1: Model parameters

Panel A: Heston model parameters

μ	σ	k	c	ρ
0.060	0.150	0.400	0.800	-0.300

Panel B: Pricing kernel parameters

α	β	γ
1.100	-2.444	1.693

Figure 2 depicts empirical pricing kernels and the exact pricing kernel by Eq.(4.3). The left panels of Figure 2 show empirical pricing kernels estimated by the PKE estimation with different sizes of sample returns, while the right panels show them estimated by the standard method. The dotted lines denote empirical pricing kernels and the solid lines denote the exact pricing kernel. There are ten dotted lines of empirical pricing kernels on each panel, which are estimated from same size, but independent sample sets. On the panels with 100 sample size, empirical pricing kernels are oscillating nevertheless the exact pricing kernel is U-shaped. This concludes that empirical pricing kernels with less than 100 sample returns cannot converge to exact one without depending on a method. Some lines of empirical pricing kernels in the panels look tilde-shape like empirical observations reported in past literature. In fact, it is not until 10,000 sample size that empirical pricing kernels approach to exact one. However, if we want to use 10,000 non-overlapping monthly returns in practice, we have to collect historical data for over 833 years!

Disappointedly, Figure 2 indicates that the PKE estimation is not superior to the standard method. In the following, we will look into the cause of the non-superiority. The left panels of Figure 3 depict empirical cumulative distribution functions estimated by the kernel density estimation with the exact one computed by formula (A.5). Recall that the empirical cumulative distribution functions are used for the PKE estimation. The right panels of Figure 3 exhibit empirical probability density functions estimated by the kernel density estimation with the exact one computed by Lévy's inversion formula. The empirical probability density functions are used for the standard method. In the same manner as Figure 2, there are ten dotted lines denoting empirical probability distributions and one solid line denoting the exact probability distribution on each panel. Following to the theoretical result discussed in Appendix B, Figure 3 illustrates that empirical cumulative distribution functions converge to exact one faster than empirical density functions. However, empirical pricing kernels estimated by the PKE estimation seem not to receive any benefit from the faster convergence rate.

Next, we examine accuracy of implied risk-neutral probability distributions. Figure 4 plots the implied risk-neutral cumulative distribution function plotted by circles and the implied risk-neutral probability density function plotted by asterisks, which are computed by formulas (C.1) and (C.2), respectively. These are estimated from the implied volatility curve in Figure 1. Although the implied risk-neutral cumulative distribution function used for the PKE estimation is perfectly coincident with exact one denoted by the solid line, the implied risk-neutral probability density function used for the standard method slightly deviates from exact one denoted by the dotted line. In fact, the case of 100 million sample in Figure 5 illustrates that the empirical pricing kernel made by the standard method denoted by the dotted line deviates from the exact pricing kernel denoted by the solid line parallel to the deviation of the implied risk-neutral probability density function. In contrast, the empirical pricing kernel estimated by the PKE estimation denoted by the dash line perfectly fits exact one. Consequently, the PKE estimation is advantageous in terms of the curse of differentiation.

In conclusion, the practical advantage of the PKE estimation becomes apparent only if an incalculable number of sample returns are available. However, as an important implication of the simulation test, it can be said that it is essentially difficult to estimate empirical pricing kernels in any manner as long as the kernel density estimation with a limited number of sample data are used for. Shefrin (2008) pointed out as follows: The subjective density function of a representative investor can be represented as a wealth-weighted convex combination of the individual investors' subjective density functions. If the individual investors have heterogeneous beliefs and even if each of their subjective density functions portrays a pure bell curve, the shape of the representative investor's subjective density function may be distorted like the empirical density functions on the first right panel of Figure 3. Shefrin (2008) concluded that the distorted subjective density function generates an oscillating pricing kernel. His insight holds true for any pricing kernel estimation methods with the kernel density estimation. The use of the kernel estimation method implicitly assumes heterogeneous beliefs. That is to say, the number of sample returns in the kernel estimation method effectively corresponds the number of heterogeneous investors, because an estimated density function is composed of an equal-weighted convex combination of the values of the kernel density associated with each sample returns. Thereby, a limited number of sample returns could induce a distorted empirical density function nevertheless the exact density function portrays a pure bell curve. Such an estimation result generates an oscillating empirical pricing kernel. See also Appendix B.

5 Subjective Statistics Representation

Unfortunately, the previous section reveals that estimating empirical pricing kernels by the PKE estimation may not be necessarily advantageous. However, we attempt another application that we derive subjective probability distributions from the PKE with a given suitable reciprocal kernel. As will be shown, subjective fundamental statistics can also be measured in a similar manner. For modeling a suitable reciprocal kernel, suppose that it is given by

$$g(x) = R_f \frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_x = R_f \frac{g_0(x)}{\mathbb{E}_*\left[g_0(S_T)\right]}, \tag{5.1}$$

where $g_0(x)$ is a twice differentiable and strictly positive function of $x > 0$. It would be convenient to directly model the reciprocal kernel rather than the pricing kernel itself. For example, to treat the pricing kernel defined in Eq.(4.3), we have to know the subjective probability on the asset price in advance for computing the denominator on the right side of Eq.(4.3). In contrast, when treating reciprocal kernel (5.1), we do not need to know any information about the subjective probability. Instead, we can use option prices or implied volatilities observed in markets to determine reciprocal kernel (5.1), because it is represented as

$$g(x) = Gg_0(x),$$

where G is the reciprocal of the price of payoff $g_0(S_T)$. That is, the constant G is given by

$$\frac{1}{G} := \frac{1}{R_f}\mathbb{E}_*\left[g_0(S_T)\right] = \frac{1}{R_f}g_0(\kappa) + \int_0^\kappa g_0''(K)P(T,K)dK + \int_\kappa^\infty g_0''(K)C(T,K)dK,$$

where $\kappa := R_f S_0$ denotes the forward price of the asset maturing at time T and $C(T, K)$ denotes the current price of a call option maturing at time T with strike price K. Therefore, reciprocal kernel (5.1) is model-free for both the subjective and the risk-neutral probability distributions on the asset price. The functional form of $g_0(x)$ can be chosen from a wide variety of functions.

For the rest of this section, we presume that the reciprocal kernel $g(x)$ is given in any form. Section 5.2 provides a general form of $g_0(x)$ connecting with absolute risk aversion. In the next section, we will develop the explicit representation of reciprocal kernel (5.1) consistent with hyperbolic absolute risk aversion (HARA for short). Some formulas for measuring subjective statistics are provided in Sections 5.1 and 5.3. All the results below can be regarded as corollaries of Theorem 1.

5.1 Subjective Moments

The next proposition is a simple application of Lemma 1, but a convenient tool for measuring subjective fundamental statistics implied from observed option prices.

Proposition 1 (Subjective Moment Formula) *Let $f(x)$ be a twice differentiable function with respect to $x > 0$. Then, we have*

$$\mathbb{E}\left[f(S_T)\right] = \frac{f(\kappa)g(\kappa)}{R_f} + \int_0^\kappa \left(f(K)g(K)\right)'' P(T,K)dK + \int_\kappa^\infty \left(f(K)g(K)\right)'' C(T,K)dK. \quad (5.2)$$

Proof of Proposition 1: Applying Eq.(3.5) to the product of the functions $f(S_T)g(S_T)$ instead of $g(S_T)$ with $x = \kappa$, we obtain Eq.(5.2) by Lemma 1. □

To obtain the subjective n-th moment of the asset price at time T, we set $f(x) = x^n$ in Proposition 1. For example, the subjective expected value of the asset price is written as

$$\begin{aligned}\mathbb{E}\left[S_T\right] \;=\; & S_0 g(\kappa) + \int_0^\kappa \left(2g'(K) + Kg''(K)\right) P(T,K)dK \\ & + \int_\kappa^\infty \left(2g'(K) + Kg''(K)\right) C(T,K)dK. \end{aligned} \quad (5.3)$$

Similarly to the subjective cumulative distribution function, subjective expected value (5.3) is priced by a static portfolio. The static portfolio is composed of a long position in $\kappa g(\kappa)$ units of the risk-free asset, a long position in $(2g'(K) + Kg''(K))dK$ units of put options at all strikes less than κ, and a long position in $(2g'(K)+Kg''(K))dK$ units of call options at all strikes larger than κ. The subjective expected value in Eq.(5.3) is affected by not only the reciprocal pricing itself, but also its slope and curvature. When the representative investor in the preliminary example of Section 2 is risk-neutral, the right side of Eq.(5.3) equals the forward price of the asset as expected.

5.2 Absolute Risk Aversion

Suppose that the pricing kernel is represented as the marginal utility form in Eq.(2.4) in Section 2. In this case, the absolute risk aversion of the representative investor can be written as

$$ARA(x) = \frac{g'(x)}{g(x)}. \tag{5.4}$$

Consequently, the next proposition, which is immediately derived from Eq.(5.4), shows a general formula to obtain the functional form of $g_0(x)$ in Eq.(5.1) consistent with arbitrary absolute risk aversion.

Proposition 2 (Reciprocal Kernel) *Suppose that the pricing kernel is represented as the marginal utility form in Eq.*(2.4)*. Then, the function $g_0(x)$ in Eq.*(5.1) *has the form*

$$g_0(x) = \exp\left\{\int^x ARA(y)dy\right\}. \tag{5.5}$$

Proof of Proposition 2: Eq.(5.4) leads to the homogeneous linear ordinary differential equation of the first order for $g_0(x)$

$$g_0'(x) = ARA(x)g_0(x).$$

The general solution to the above equation is given by

$$g_0(x) = C\exp\left\{\int^x ARA(y)dy\right\}, \tag{5.6}$$

where C is some constant. Because of Eq.(5.1), we can put $C = 1$ without loss of generality. □

Note that the projected pricing kernel $\mathcal{M}(x)$ can be written as the reciprocal of general solution (5.6). However, we have to know the subjective probability distribution in advance to determine the constant C. In contrast, the static portfolio representation of the constant G is crucial for the reciprocal kernel in Eq.(5.1).

As a simple example of Proposition 2, by putting the absolute risk aversion constant, the functional form of the reciprocal kernel in Eq.(5.1) is obtained consistent with an exponential utility function. Proposition 2 also indicates that, under the assumption that the pricing kernel is given by the marginal utility form in Eq.(2.4), we can rewrite the PKE and the subjective moment formula in Proposition 1 in terms of the absolute risk aversion instead of the reciprocal kernel.

If the subjective and the risk-neutral cumulative distribution functions are absolutely continuous and their density functions are differentiable, then absolute risk aversion (5.4) can be rewritten as

$$ARA(x) = \frac{\phi'(x)}{\phi(x)} - \frac{\phi_*'(x)}{\phi_*(x)}. \tag{5.7}$$

Expression (5.7) originally introduced by Leland (1980) has often been used in past literature. However, it is not our objective because there are not only the probability density functions, but also their derivatives in expression (5.7). As shown in Section 4, both the subjective and the risk-neutral density functions are not robust in empirical estimations. Thus, the numerical differentiation of such density functions would be unstable in nature. As a result, there is little hope of obtaining accurate values of Eq.(5.7).

5.3 Kullback-Leibler Divergence

The Kullback-Leibler divergence also known as the *relative entropy* is a statistic to measure difference between two probability distributions. The definition of the Kullback-Leibler divergence between the risk-neutral and the subjective probability distributions on S_T is

$$KL(\mathbb{Q}\|\,\mathbb{P}) := -\int \log\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{S_T}\right) d\mathbb{Q}. \tag{5.8}$$

It is always non-negative and it takes zero if and only if the two probability distributions are identical. Recall that the Kullback-Leibler divergence is an asymmetric measure and thus does not qualify as a distance.

The following proposition demonstrates that the Kullback-Leibler divergence between the risk-neutral and the subjective probability distributions is priced by a static portfolio including plain vanilla options.

Proposition 3 (Kullback-Leibler Divergence) *Suppose that the pricing kernel is represented as the marginal utility form in Eq.(2.4). Then, the Kullback-Leibler divergence between the risk-neutral and the subjective probability distributions is given by*

$$\begin{aligned} KL(\mathbb{Q}\|\,\mathbb{P}) &= \log R_f - \log g(\kappa) \\ &\quad - R_f\left\{\int_0^\kappa ARA'(K)P(T,K)dK + \int_\kappa^\infty ARA'(K)C(T,K)dK\right\}. \end{aligned} \tag{5.9}$$

Proof of Proposition 3: By the definition of Kullback-Leibler divergence (5.8), we have

$$KL(\mathbb{Q}\|\,\mathbb{P}) = -\mathbb{E}_*\left[\log\left(\frac{g(S_T)}{R_f}\right)\right] = \log R_f - \mathbb{E}_*\left[\log g(S_T)\right].$$

Applying Eq.(3.5) to $\log g(S_T)$ instead of $g(S_T)$ with $x = \kappa$ and noting from Eq.(5.4) that

$$(\log g(x))' = ARA(x),$$

yield

$$\frac{1}{R_f}\mathbb{E}_*\left[\log g(S_T)\right] = \frac{\log g(\kappa)}{R_f} + \int_0^\kappa ARA'(K)P(T,K)dK + \int_\kappa^\infty ARA'(K)C(T,K)dK.$$

□

Amounts of the options in the static portfolio pricing the Kullback-Leibler divergence depend on the slope of the absolute risk aversion. When the representative investor is risk-neutral, the slope of his/her absolute risk aversion is zero and $g(x) = R_f$ for any $x > 0$. In this case, the Kullback-Leibler divergence is zero, that is, the risk-neutral probability distribution is identical to the subjective one.

6 Numerical Example

This section presents a numerical example in which given an implied volatility curve with one-year maturity and one-year interest rate, we estimate subjective probabilities and subjective fundamental statistics about the one year later value of the market portfolio.

Suppose that S_t is the time-t value of the market portfolio and $g_0(x)$ in Eq.(5.1) is given by

$$g_0(x) := \left(\frac{\alpha x}{1-\gamma} + \beta \right)^{1-\gamma}, \tag{6.1}$$

where α, β, and γ are some constants. From relation (5.4), the risk tolerance, which is defined as the reciprocal of the absolute risk aversion, has the form

$$RT(x) = \frac{g(x)}{g'(x)} = \frac{x}{1-\gamma} + \frac{\beta}{\alpha}. \tag{6.2}$$

Therefore, plugging $g_0(x)$ in Eq.(6.1) into reciprocal kernel (5.1) means that the representative investor has a HARA utility over the market portfolio. That is, his/her utility function is given by

$$U(x) = \frac{1-\gamma}{\gamma} \left(\frac{\alpha x}{1-\gamma} + \beta \right)^{\gamma}.$$

Recall that if $\gamma < 1$ and $\alpha = 1-\gamma$, the investor has a shifted power utility function. When $\alpha = 0$ and $\gamma = 0$, the investor has a shifted log utility function. The relative risk aversion is increasing in x if $\beta > 0$ and decreasing in x if $\beta < 0$. Putting $\beta = 0$ means constant relative risk aversion. When $\gamma = 1$, the representative investor is risk-neutral.

The one-year implied volatility curve of the market portfolio that is regarded as available market information is depicted in Figure 6. Another market information given there is one-year risk-free interest rate, whose value is set as $r := \log R_f = 2.5\%$. The implied volatility curve is generated by a risk-neutral Heston's stochastic volatility model, which is described by Eq.(4.1) with drift term r instead of μ because of risk-neutral modeling. The values of the Heston model parameters are listed in Table 2.

Table 2: Heston model parameters

r	σ	k	c	ρ
0.025	0.150	0.400	0.800	-0.400

Table 3 exhibits the subjective probabilities derived from the PKE associated with the implied volatility curve in Figure 6 and the reciprocal kernel in Eq.(6.1). We set the values of the utility parameters as $\alpha = 1-\gamma$, β = -0.2, 0, or 0.2, and γ = 1, 0, -1, $\ldots$, or -4. Risk tolerance (6.2) indicates that the more negative parameter γ is, the more risk averse is the investor. The first row of Table 3 displays the risk-neutral probabilities ($\gamma = 0$) as a benchmark. As a whole, with increasing risk aversion, the investor underestimates his/her subjective probabilities on the event that the one-year later value of the market portfolio is lower than a certain value.

We first make a comment on the constant relative risk aversion case ($\beta = 0$). The subjective probability that the value of the market portfolio falls below 0.9 is 15.9% when the investor has a log utility function ($\gamma = 0$), while it is only 5.5% when γ = -4 that means the investor has a power utility function with constant relative risk aversion of 5. Next, we focus on the decreasing relative risk aversion case (β = -0.2). The subjective probabilities there are more underestimated than the constant relative aversion case. Conversely, the subjective probabilities in the increasing relative risk aversion case ($\beta = 0.2$) are more overestimated than the constant relative risk aversion case.

Next, we investigate subjective fundamental statistics of the one-year later log-return on the market portfolio. We begin by computing the subjective n-th central moment of the log-return

Table 3: Subjective probabilities

β	γ	$S_1 \leq 0.9$	$S_1 \leq 1.0$	$S_1 \leq 1.1$	$S_1 \leq 1.2$	$S_1 \leq 1.3$
–	1	0.200	0.432	0.696	0.881	0.964
-0.2	0	0.149	0.360	0.632	0.844	0.949
	-1	0.109	0.296	0.566	0.802	0.929
	-2	0.078	0.238	0.500	0.754	0.905
	-3	0.055	0.189	0.435	0.701	0.875
	-4	0.038	0.147	0.373	0.644	0.840
0	0	0.159	0.374	0.644	0.852	0.952
	-1	0.125	0.321	0.591	0.818	0.937
	-2	0.096	0.271	0.537	0.780	0.918
	-3	0.073	0.227	0.483	0.739	0.896
	-4	0.055	0.187	0.430	0.694	0.870
0.2	0	0.166	0.384	0.653	0.857	0.954
	-1	0.136	0.338	0.608	0.829	0.941
	-2	0.110	0.295	0.563	0.798	0.927
	-3	0.088	0.255	0.517	0.764	0.909
	-4	0.070	0.219	0.471	0.727	0.889

to apply Proposition 1 to the function

$$f(x) := [\log x - m]^n ,$$

where m is the first moment if $n \geq 2$ and zero if $n = 1$. Based on up to the fourth central moment, we then compute subjective perspectives of mean (Mean), standard deviation (StdDev), skewness (Skew), and excess kurtosis (Kurt) of the log-return. In addition, we also measure the Kullback-Leibler divergence (KLD) between the risk-neutral and the subjective probabilities on the market portfolio value by formula (5.9).

Table 4 exhibits the subjective fundamental statistics and the Kullback-Leibler divergence. The risk-neutral distribution ($\gamma = 1$) has the mean of 1.4%[1], the standard deviation of 15.2%, strongly negative skewness (-0.445), and larger excess kurtosis (0.624). These results are regarded as benchmark statistics. We first discuss the constant relative risk aversion case ($\beta = 0$). As expected, the investor with higher risk aversion has his/her perspective of the log-return distribution with larger mean and smaller standard deviation. With increasing risk aversion, the subjective negative skewness and the subjective excess kurtosis are gradually mitigated. As a result, the more risk averse is, the larger is the Kullback-Leibler divergence. When γ = -2, which means that the investor has a power utility function over the market portfolio with constant relative risk aversion of 3, the subjective mean is 7.7% that seems to be a plausible level in comparison with historical average of log-returns on the S&P 500. Next, paying attention to the decreasing relative risk aversion case (β = -0.2), we notice that the subjective fundamental statistics are more sensitive to parameter γ than the constant relative risk aversion case. Conversely, these statistics are less sensitive in the case of the increasing relative risk aversion (β = 0.2).

[1] Note that the risk-neutral mean of the log-return is not $r = 0.025$, but $r - \sigma^2/2 = 0.01375$ more precisely.

7 Conclusion

This paper presents the pricing kernel equation and its applications. It is demonstrated that the PKE does not only offer a general formula for the theoretical relation between the subjective and the risk-neutral cumulative distribution functions, but also it could be a versatile tool for various applications. The applications described in this paper are just a part of them. Other conceivable applications are as follows.

1. **To estimate the probability weighting function:** The probability weighting function is an aspect of the cumulative prospect theory introduced by Tversky & Kahneman (1992) and defined as distortion of subjective probabilities to objective probabilities. Compering the subjective probability distribution implied by the PKE with an empirical distribution regarded as the objective probability distribution, one might estimate the probability weighting function from market data.

2. **To construct financial risk measurement:** Generally speaking, standard risk measurement such as Value at Risk is a lagging indicator due to the use of historical data. The subjective probability distribution estimated by the PKE from real time option prices could be used for financial risk measurement as a forward-looking indicator.

3. **To develop new trading strategies:** We suggest new trading strategies dealing with the subjective probability distribution. For example, the PKE makes it possible to robustly replicate an option depending on a subjective probability like a digital put option. As shown in Table 3, its price that equals the subjective probability is cheaper than the price of the digital put option. These payoffs are the same when they are in the money, but different when they are out of the money.

4. **To produce new indices:** Like the VIX, the subjective mean computed in Section 6 might become a new index measuring market sentiment as well as the subjective standard deviation, skewness, and kurtosis. These indices represent bullish, bearish, fear, or conscious of rare events for the market.

The reciprocal kernel itself is also thought of an interesting concept in spite of the very simple definition. The reciprocal kernel makes it possible to describe the pricing kernel more flexibly in a tractable manner. In this paper, we apply it to specify the pricing kernel consistent with the hyperbolic absolute risk aversion utility. In future research, we would like to use the reciprocal kernel for modeling non-standard pricing kernels including the tilde-shaped pricing kernel generated by the ambiguity aversion model of Klibanoff et al. (2005).

Finally, we acknowledge that the PKE is essentially for single-period asset pricing models and it does not describe any intertemporal characteristics of the relation between the subjective and the risk-neutral probability distributions. A natural direction for future research is to extend the PKE to multi-period models. Although such an extension is necessary for addressing time-inseparable utility functions such as the habit formation by Campbell & Cochrane (1999) and the recursive utility by Epstein & Zin (1989), it might be a challenging task.

A Heston Model Analysis

The following lemma is useful to analyze Heston model (4.1).

Table 4: Subjective fundamental statistics

β	γ	Mean	StdDev	Skew	Kurt	KLD
−	1	0.014	0.152	-0.445	0.624	0.000
-0.2	0	0.041	0.145	-0.371	0.589	0.018
	-1	0.067	0.140	-0.293	0.544	0.069
	-2	0.091	0.137	-0.216	0.501	0.150
	-3	0.113	0.134	-0.140	0.461	0.260
	-4	0.135	0.133	-0.068	0.423	0.397
0	0	0.036	0.147	-0.392	0.612	0.011
	-1	0.057	0.143	-0.334	0.586	0.044
	-2	0.077	0.140	-0.273	0.556	0.097
	-3	0.096	0.138	-0.210	0.523	0.170
	-4	0.115	0.136	-0.147	0.490	0.262
0.2	0	0.032	0.148	-0.404	0.621	0.008
	-1	0.050	0.145	-0.358	0.607	0.031
	-2	0.067	0.142	-0.309	0.586	0.068
	-3	0.084	0.140	-0.257	0.561	0.120
	-4	0.101	0.138	-0.203	0.534	0.186

Figure 1: Implied volatility for Section 4

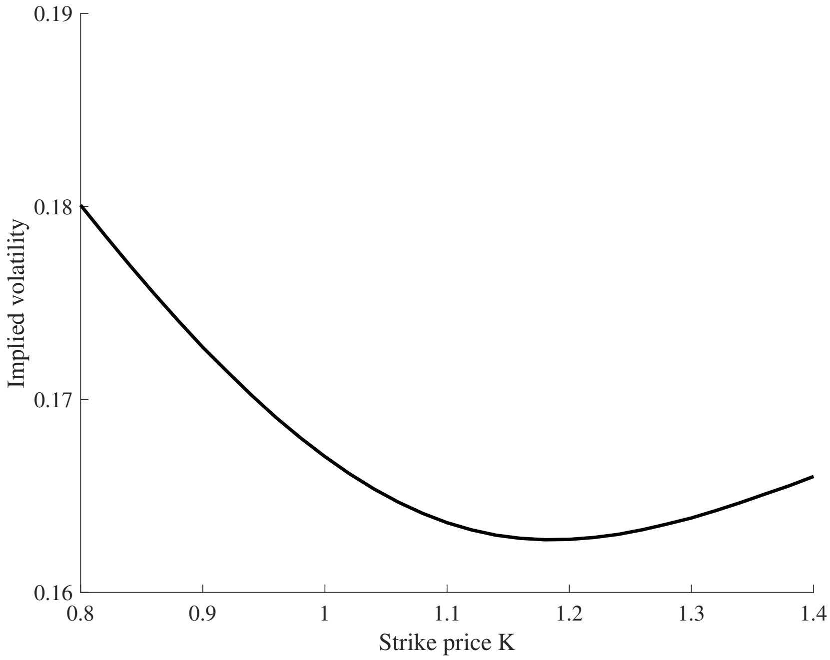


Figure 2: Empirical pricing kernels

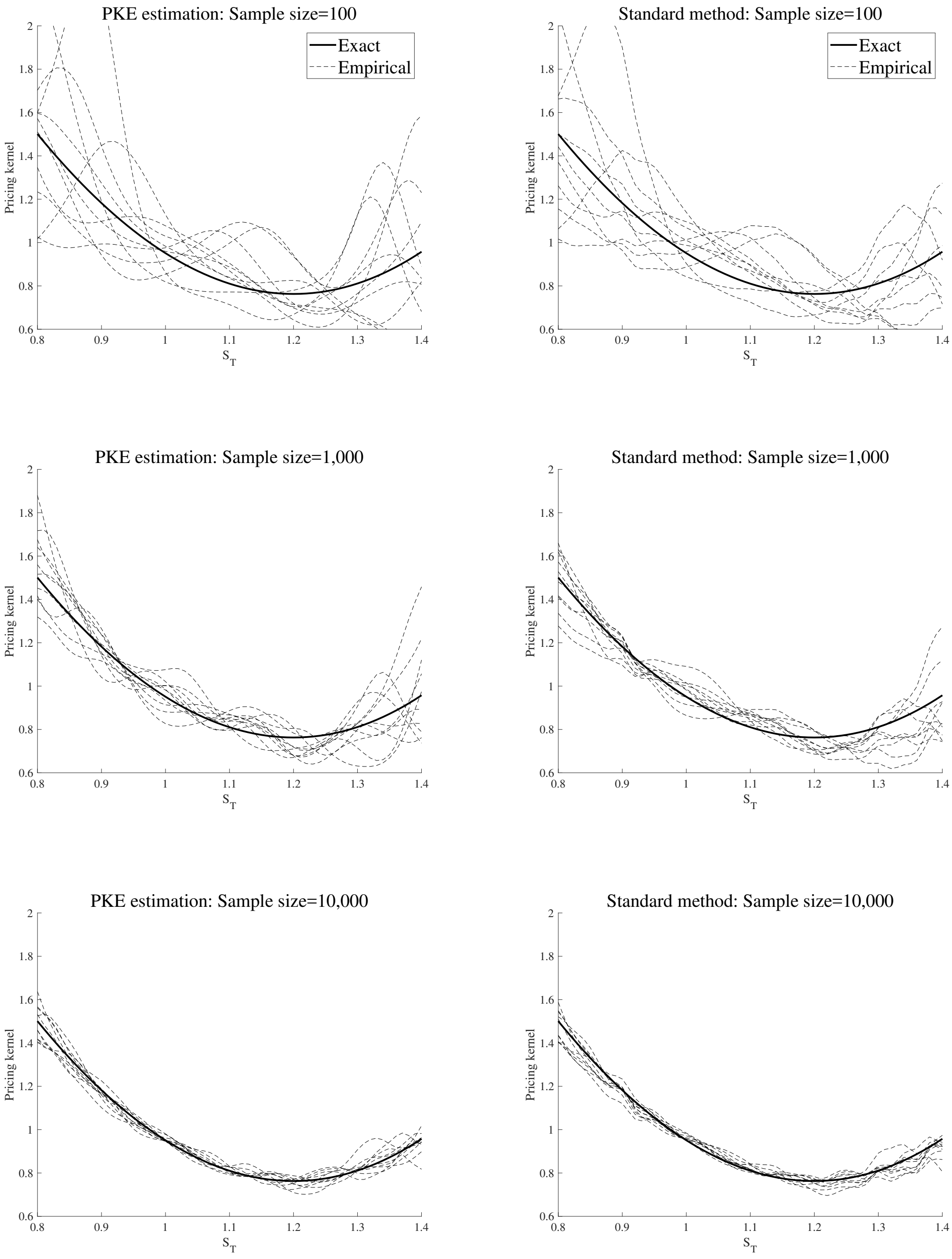


Figure 3: Empirical probability distribution

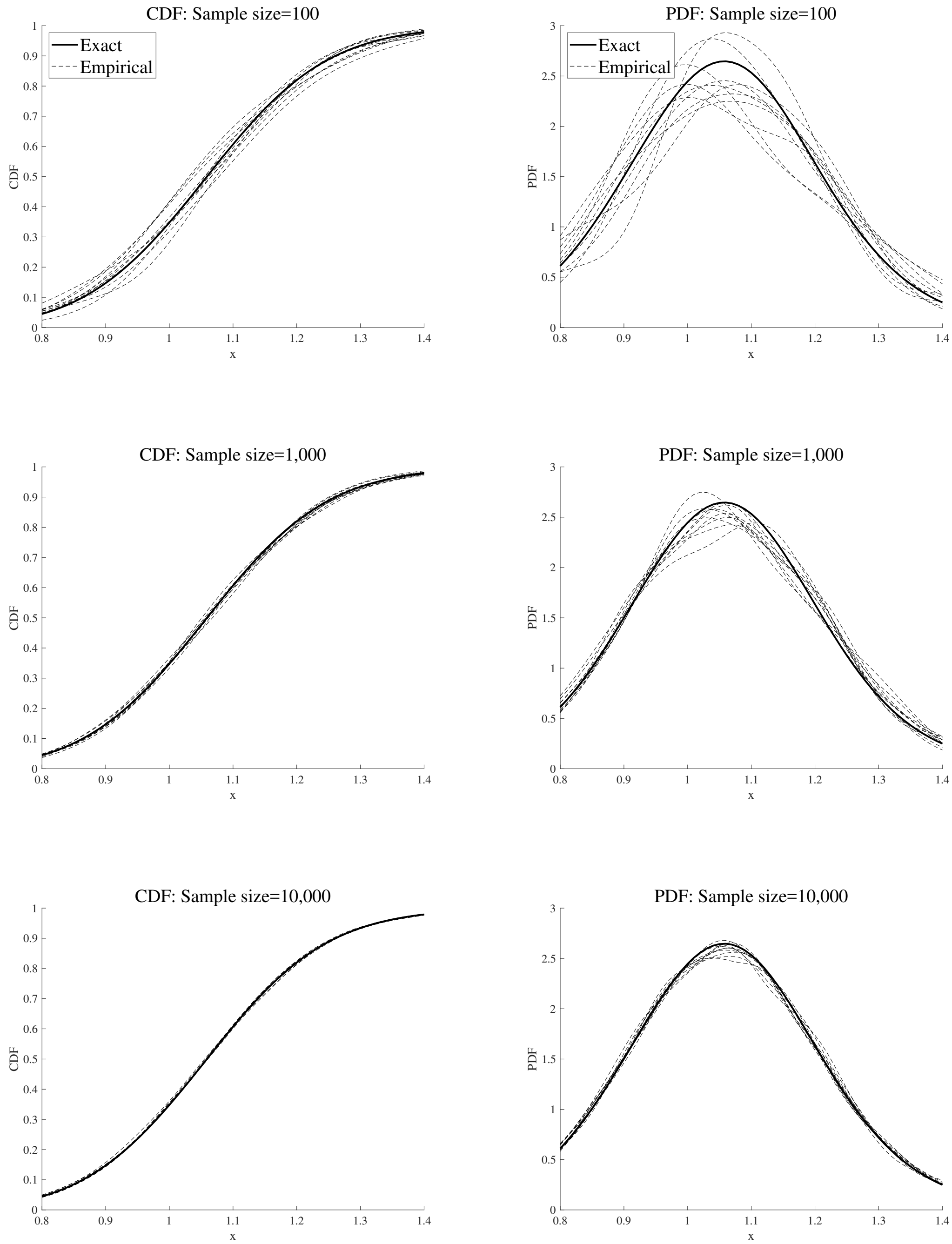


Figure 4: Risk-neutral probability distribution

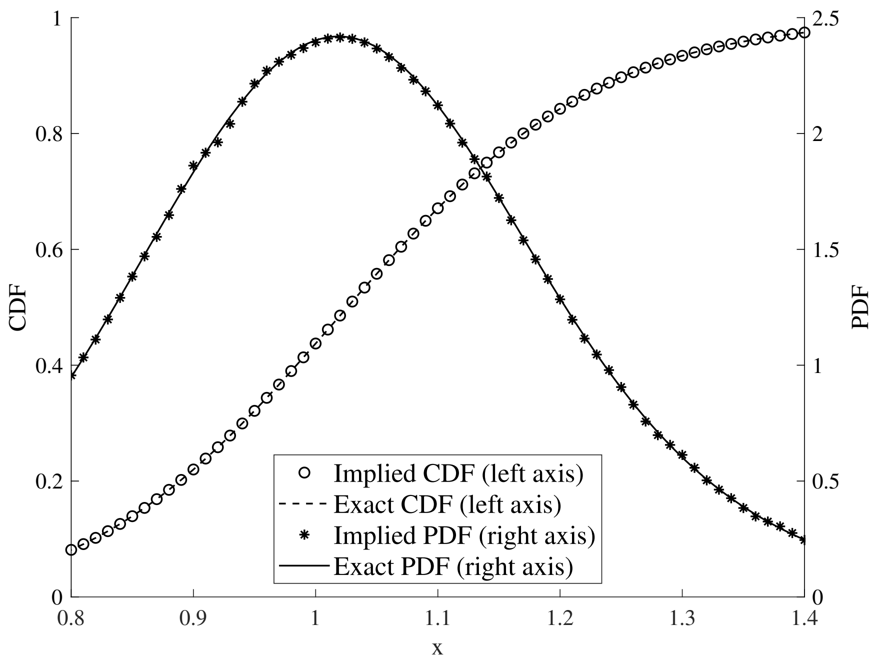


Figure 5: Empirical pricing kernel with 100 million samples

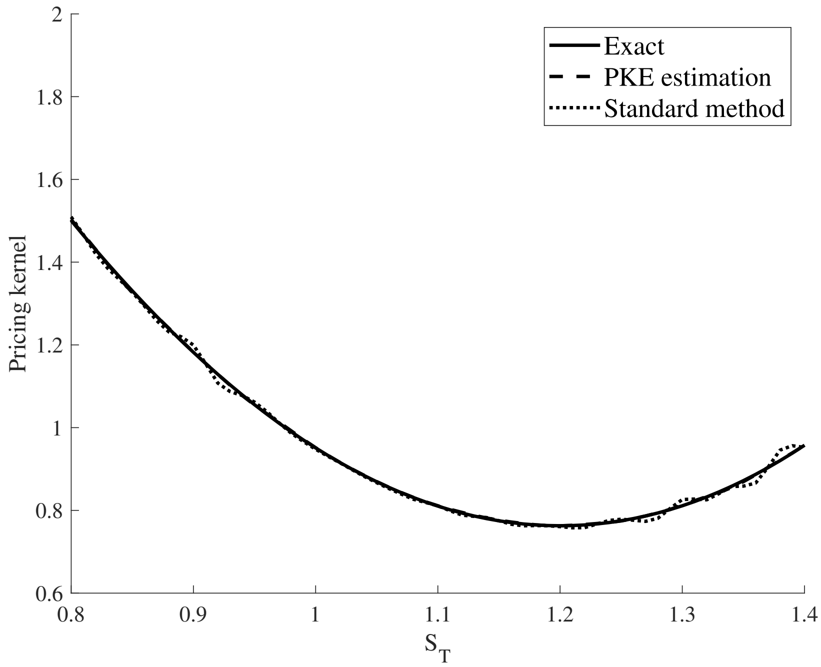
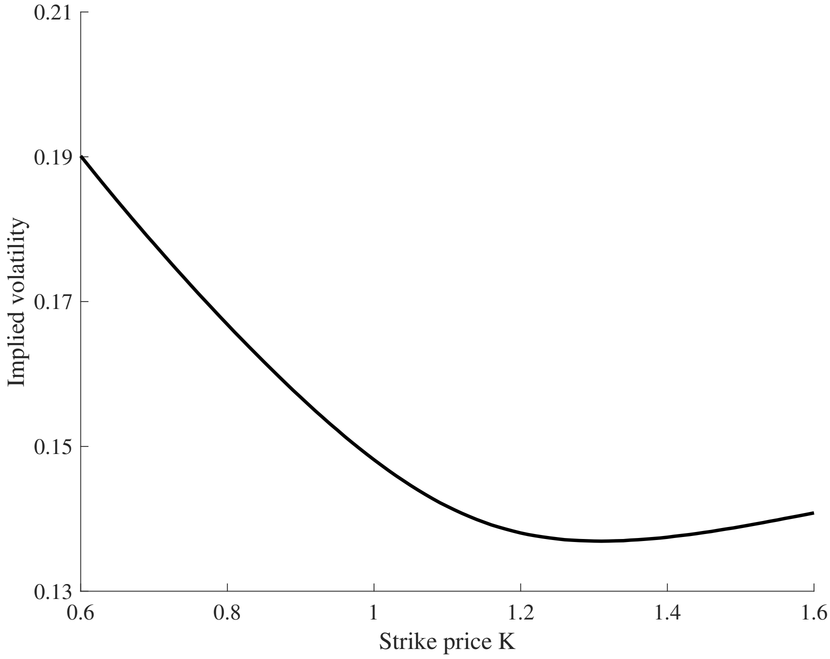


Figure 6: Implied volatility for Section 6



Lemma 2 *Define*

$$\xi(T,x) := \mathbb{E}\left[\exp\left\{\left(\mu T - \frac{1}{2}\sigma^2 \int_0^T v_t dt + \sigma \int_0^T \sqrt{v_t} dB_t^1\right) x\right\}\right].$$

Then, it is represented as $\xi(T,x) = \exp\{x\mu T + \Lambda_1(T,x) + \Lambda_2(T,x)\}$, *where*

$$\Lambda_1(T,x) := -\frac{2k}{c^2}\log\lambda_1(T), \qquad \text{and} \qquad \Lambda_2(T,x) := \frac{2}{c^2}\frac{\lambda_2(T)}{\lambda_1(T)}.$$

Here, putting $k_1 := x\sigma c\rho - k$ *and* $k_2 := \frac{1}{2}\sigma^2 c^2(x^2 - x)$, *if* $k_1^2 - 2k_2 \neq 0$,

$$\lambda_1(T) := \frac{c_+}{2\eta}e^{-c_- T} - \frac{c_-}{2\eta}e^{-c_+ T}, \quad \text{and} \quad \lambda_2(T) := \frac{c_+ c_-}{2\eta}\left(e^{-c_- T} - e^{-c_+ T}\right),$$

where $c_\pm := -\frac{1}{2}k_1 \pm \eta$, $\eta := \frac{1}{2}\sqrt{k_1^2 - 2k_2}$. *Otherwise,*

$$\lambda_1(T) := \left(1 - \frac{1}{2}k_1 T\right)e^{\frac{1}{2}k_1 T}, \quad \text{and} \quad \lambda_2(T) := \frac{1}{2}k_1 e^{\frac{1}{2}k_1 T} - \frac{1}{2}k_1\lambda_1(T).$$

Proof of Lemma 2: Define the new probability measure $\mathbb{P}(x)$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{P}(x)}{d\mathbb{P}} := \exp\left\{x\sigma\int_0^T \sqrt{v_t} dB_t^1 - \frac{1}{2}x^2\sigma^2\int_0^T v_t dt\right\}.$$

Then, we have

$$\begin{aligned}\xi(T,x) &= e^{x\mu T}\mathbb{E}\left[\frac{d\mathbb{P}(x)}{d\mathbb{P}}\exp\left\{\frac{1}{2}x^2\sigma^2\int_0^T v_t dt - \frac{1}{2}x\sigma^2\int_0^T v_t dt\right\}\right] \\ &= e^{x\mu T}\mathbb{E}_x\left[\exp\left\{z\int_0^T v_t dt\right\}\right], \qquad (A.1)\end{aligned}$$

where $\mathbb{E}_x[\,\cdot\,]$ is the expectation operator under $\mathbb{P}(x)$ and $z := \frac{1}{2}\sigma^2(x^2-x)$. Following to the Girsanov theorem, $B_t^x := B_t^1 - x\sigma\int_0^t\sqrt{v_u}du$ is a standard Brownian motion under $\mathbb{P}(x)$. Therefore, the variance process of the Heston model under $\mathbb{P}(x)$ is governed by

$$dv_t = (k + [x\sigma c\rho - k]v_t)dt + c\sqrt{v_t}\left(\rho dB_t^x + \sqrt{1-\rho^2}dB_t^2\right). \qquad (A.2)$$

Because variance process (A.2) is an affine process, we can apply Proposition 2 in Duffie, et al (2000) to Eq.(A.1). As a result, we have $\xi(T,x) = \exp\{\Lambda_1(T,x)+\Lambda_2(T,x)\}$, where $\Lambda_1(T,x) := \alpha_T(0)$ and $\Lambda_2(T,x) := \beta_T(0)$ such that the functions $\alpha_T(t)$ and $\beta_T(t)$ satisfy the system of the ODEs

$$\begin{aligned}&\frac{d}{dt}\beta_T(t) = -z - k_1\beta_T(t) - \frac{1}{2}c^2\beta_T(t)^2, \\ &\frac{d}{dt}\alpha_T(t) = -k\beta_T(t),\end{aligned}$$

with the boundary conditions $\alpha_T(T) = 0$ and $\beta_T(T) = 0$. The derivation of the solution to the system of the ODEs above can be found in Appendix of Umezawa & Yamazaki (2014) for instance. □

Making use of the asset pricing formula $S_0 = \mathbb{E}[\mathcal{M}S_T]$ with the normalized initial value $S_0 = 1$, substituting pricing kernel (4.3) into it, and applying Lemma 2, we obtain the representation of the yield to maturity

$$r(T) = \log\left[\frac{\xi(T,2\alpha+1)+\beta\xi(T,\alpha+1)+\gamma\xi(T,1)}{\xi(T,2\alpha)+\beta\xi(T,\alpha)+\gamma}\right]. \qquad (A.3)$$

Analogously, the projected pricing kernel is given by

$$\mathcal{M}(x) = \frac{x^{2\alpha}+\beta x^{\alpha}+\gamma}{\xi(T,2\alpha+1)+\beta\xi(T,\alpha+1)+\gamma\xi(T,1)}. \qquad (A.4)$$

Projected pricing kernel (A.4) is a quadratic function of x^{α}. To generate a strictly positive U-shaped pricing kernel, we restrict the parameters such that $\alpha > 0$, $\beta = -2x_{\min}^{\alpha}$, and $\gamma > x_{\min}^{2\alpha}$, where $\mathcal{M}(x)$ takes the minimum value at the point $x_{\min}$. According to the restriction, we set $x_{\min} = 1.2$, $\alpha = 1.1$, $\gamma = x_{\min}^{2\alpha} + 0.2$ in Panel B of Table 1.

A.1 Subjective Cumulative Distribution Function

The characteristic function of the log-price of the asset under the subjective probability measure $\mathbb{P}$ is defined as

$$\varphi(\theta) := \mathbb{E}\left[\exp\{i\theta\log S_T\}\right], \qquad \text{for} \quad \theta\in\mathcal{D},$$

where $\mathcal{D}$ denotes a subset of $\mathbf{C}$ such that the function $\varphi(\theta)$ is well defined on $\mathcal{D}$. Making use of Lemma 2, we have

$$\varphi(\theta) = \xi(T, i\theta).$$

Applying Gil-Pelaez's inversion formula, the subjective cumulative distribution function of S_T can be represented as

$$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\theta}\mathfrak{Im}\left\{x^{-i\theta}\varphi(\theta)\right\} d\theta. \tag{A.5}$$

A.2 Option Pricing

The risk-neutral characteristic function of the log-price of the asset defined as

$$\varphi_*(\theta) := \mathbb{E}_*\left[\exp\left\{i\theta \log S_T\right\}\right], \qquad \text{for} \quad \theta \in \mathcal{D}_* \subset \mathbf{C},$$

is given by

$$\begin{aligned}\varphi_*(\theta) &= \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\exp\left\{i\theta \log S_T\right\}\right] \\ &= \frac{\xi(T, 2\alpha + i\theta) + \beta\xi(T, \alpha + i\theta) + \gamma\xi(T, i\theta)}{\xi(T, 2\alpha) + \beta\xi(T, \alpha) + \gamma},\end{aligned} \tag{A.6}$$

where $\mathcal{D}_*$ is a subset of $\mathbf{C}$ such that the function $\varphi_*(\theta)$ is well defined on $\mathcal{D}_*$. In the second equality of Eq.(A.6), we substitute Eq.(4.2) into it and use Lemma 2.

The price of a call option written on the asset with maturity T and strike price K denoted by $C(T, K)$ can be calculated by

$$C(T, K) = \mathbb{E}_*\left[\frac{1}{R_f}\left(S_T - K\right)^+\right] = e^{-r(T)}\left\{\mathfrak{F}^{-1}[\zeta](\log K) + \left(e^{r(T)} - K\right)^+\right\}, \tag{A.7}$$

where $\mathfrak{F}^{-1}[\zeta](x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\zeta(\theta)d\theta$ is the inverse Fourier transform of the complex-valued function $\zeta(\theta)$ defined by

$$\zeta(\theta) := \frac{\varphi_*(\theta - i) - e^{(i\theta+1)r(T)}}{i\theta(i\theta + 1)}.$$

The derivation of formula (A.7) can be found in Appendix A.1 of Yamazaki (2018) for example. The implied volatility curve shown in Figure 1 is obtained by the standard method computing the implied volatilities from call option prices given by formula (A.7).

B Empirical Probability Estimation

Let $X_1, X_2, \ldots, X_n$ be independent samples drawn from the probability distribution with an unknown density function $\phi(x)$ that is twice differentiable. The kernel density estimator of the density function is defined by

$$\hat{\phi}(x) := \frac{1}{nh}\sum_{i=1}^{n}\mathcal{K}\left(\frac{X_i - x}{h}\right), \tag{B.1}$$

where $h > 0$ is a bandwidth and $\mathcal{K}(\,\cdot\,)$ is a kernel function satisfying some conditions to ensure that $\hat{\phi}(x)$ is a probability density function. MISE (mean integrated squared error) of the kernel density estimator $\hat{\phi}(x)$ can be written as

$$\text{MISE}\left[\hat{\phi}(x)\right] := \int \mathbb{E}\left[\left(\hat{\phi}(x) - \phi(x)\right)^2\right] dx = a_1 h^4 + \frac{a_2}{nh} + o\left(h^4 + (nh)^{-1}\right), \tag{B.2}$$

where a_1 and a_2 are some constants. To minimize MISE (B.2), we should chose the bandwidth $h = a_3 n^{-1/5}$, where a_3 is some positive constant. Therefore, the convergence rate of MISE (B.2) with the optimal bandwidth is $n^{-4/5}$. If the standard normal density function is chosen as the kernel function, the optimal bandwidth is $h \approx 1.06\hat{\sigma}n^{-1/5}$, where $\hat{\sigma}$ is the standard deviation of the samples.

On the other hand, the estimator of the cumulative distribution function $F(x)$ is defined by

$$\hat{F}(x) := \int_{-\infty}^{x} \hat{\phi}(y)dy = \frac{1}{nh}\sum_{i=1}^{n} \mathcal{G}\left(\frac{X_i - x}{h}\right), \tag{B.3}$$

where $\mathcal{G}(x) := \int_{-\infty}^{x} \mathcal{K}(y)dy$. The MISE of the estimator $\hat{F}(x)$ can be written as

$$\begin{aligned}\text{MISE}\left[\hat{F}(x)\right] &:= \int \mathbb{E}\left[\left(\hat{F}(x) - F(x)\right)^2\right] dx \\ &= b_0 n^{-1} + b_1 h n^{-1} + b_2 h^4 + o\left(h^4 + hn^{-1}\right),\end{aligned} \tag{B.4}$$

where b_0, b_1, and b_2 are some constants. To minimize MISE (B.4), we should chose the bandwidth $h = b_3 n^{-1/3}$, where b_3 is some positive constant. Therefore, the convergence rate of MISE (B.4) with the optimal bandwidth is n^{-1}. It is worthwhile noting that the convergence speed of estimator (B.3) is faster than that of kernel density estimator (B.1). If the standard normal density function is chosen as the kernel function, the optimal bandwidth is $h \approx 1.59\hat{\sigma}n^{-1/3}$.

The standard method including Jackwerth (2000) uses kernel density estimator (B.1) with the standard normal density function as the kernel function to estimate empirical pricing kernels. On the other hand, we use cumulative distribution estimator (B.3) to estimate empirical cumulative distribution functions for the PKE estimation.

C Implied Risk-Neutral Probability

Given an implied volatility function $\nu(K)$ of strike price K, we can estimate the risk-neutral cumulative distribution function of the asset price. It is the first order derivative of the Black-Scholes put option price with respect to strike price after multiplying by the gross rate of the risk-free asset. That is, the implied risk-neutral cumulative distribution function $\hat{F}_*(x)$ is obtained by

$$\hat{F}_*(K) = R_f \frac{d}{dK} P(T, K) = \Psi(-d_2) + \nu'(K)\sqrt{T}K\psi(d_2), \tag{C.1}$$

where $\Psi(x)$ and $\psi(x)$ are the cumulative distribution function and the density function of the standard normal distribution, respectively. Here, we define

$$\begin{aligned} d_1 &:= \frac{1}{\nu(K)\sqrt{T}}\left\{\log\frac{S_0}{K} + r(T) + \frac{\nu(K)^2}{2}T\right\}, \\ d_2 &:= d_1 - \nu(K)\sqrt{T}. \end{aligned}$$

The risk-neutral density function can be calculated by differentiating twice the Black-Scholes put (or call) option price with respect to strike price after multiplying by the gross rate of the risk-free asset. That is, the implied risk-neutral density function $\hat{\phi}_*(x)$ is obtained by

$$\begin{aligned}\hat{\phi}_*(K) = R_f \frac{d^2}{dK^2} P(T, K) = \frac{\psi(d_2)}{K\nu(K)\sqrt{T}} &\left\{1 + 2K\nu'(K)\sqrt{T}d_1\right\} \\ &+ S_0\sqrt{T}\psi(d_1)\left\{\nu''(K) + \frac{(\nu'(K))^2}{\nu(K)} d_1 d_2\right\}.\end{aligned} \tag{C.2}$$

The standard method uses implied density function (C.2) to estimate empirical pricing kernels, while the PKE estimation uses implied cumulative distribution function (C.1).

D PKE Estimation

Suppose that an empirical cumulative distribution function $\hat{F}(x)$, an implied risk-neutral cumulative distribution function $\hat{F}_*(x)$, and put option prices $P(T, K)$ are given. To estimate the shape of the reciprocal kernel $g(x)$, we determine its values by a point-wisely quadratic approximation. Let $\{x_n\}_{n=1,\ldots,N}$ be estimation points of $g(x_n)$. The value of $g(x_n)$ is assumed to be approximated by $g_n(x_n)$, where $g_n(x) = a_n x^2 + b_n x + c_n$ is a quadratic polynomial with coefficients a_n, b_n, and c_n for $n = 1, \ldots, N$. To determine the three coefficients, we match the polynomial $g_n(x)$ with PKE (3.4) at three points including the estimation point x_n. That is,

$$\begin{cases} \hat{F}(x_n^+) = \dfrac{1}{R_f} g_n(x_n^+)\hat{F}_*(x_n^+) - g_n'(x_n^+)P(T, x_n^+) + \displaystyle\int_0^{x_n^+} g_n''(K)P(T, K)dK, \\ \hat{F}(x_n) = \dfrac{1}{R_f} g_n(x_n)\hat{F}_*(x_n) - g_n'(x_n)P(T, x_n) + \displaystyle\int_0^{x_n} g_n''(K)P(T, K)dK, \\ \hat{F}(x_n^-) = \dfrac{1}{R_f} g_n(x_n^-)\hat{F}_*(x_n^-) - g_n'(x_n^-)P(T, x_n^-) + \displaystyle\int_0^{x_n^-} g_n''(K)P(T, K)dK, \end{cases}$$

where $x_n^+ := x_n + \Delta$ and $x_n^- := x_n - \Delta$ with some positive constant Δ. Here, $\hat{F}(x)$ and $\hat{F}_*(x)$ denote the empirical and the implied risk-neutral cumulative distribution functions, respectively. This system of the equations can be rewritten as the matrix form

$$\begin{pmatrix} \hat{F}(x_n^+) \\ \hat{F}(x_n) \\ \hat{F}(x_n^-) \end{pmatrix} = \begin{pmatrix} A(x_n^+) & B(x_n^+) & C(x_n^+) \\ A(x_n) & B(x_n) & C(x_n) \\ A(x_n^-) & B(x_n^-) & C(x_n^-) \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \tag{D.1}$$

where

$$\begin{cases} A(x) = \dfrac{1}{R_f} x^2 \hat{F}_*(x) - 2xP(T, x) + 2\displaystyle\int_0^x P(T, K)dK, \\ B(x) = \dfrac{1}{R_f} x\hat{F}_*(x) - P(T, x), \\ C(x) = \dfrac{1}{R_f} \hat{F}_*(x). \end{cases}$$

Solving the system of the equations in matrix (D.1), we obtain the coefficients a_n, b_n, and c_n.

References

[1] Aït-Sahalia, Yacine and Lo, Andrew W. Nonparametric estimation of state-price densities implicit in financial asset prices. *Journal of Finance*, 53(2):499–547, 1998.

[2] Aït-Sahalia, Yacine and Lo, Andrew W. Nonparametric risk management and implied risk aversion. *Journal of Econometrics*, 94(1-2):9–51, 2000.

[3] Bakshi, Gurdip., Kapadia, Nikunj., and Madan, Dilip. Stock return characteristics, skew laws, and the differential pricing of individual equity options. *Review of Financial Studies*, 16(1):101–143, 2003.

[4] Bakshi, Gurdip., Madan, Dilip., and Panayotov, George. Returns of claims on the upside and the viability of U-shaped pricing kernels. *Journal of Financial Economics*, 97(1):130–154, 2010.

[5] Barberis, Nicholas and Huang, Ming. Stocks as lotteries: The implications of probability weighting for security prices. *American Economic Review*, 98(5):2066–2100, 2008.

[6] Berns, Gregory S., Capra, C Monica, Moore, Sara, and Noussair, Charles. A shocking experiment: New evidence on probability weighting and common ratio violations. *Judgment and Decision Making*, 2(4):234–242, 2007.

[7] Campbell, John Y and Cochrane, John H. By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy*, 107(2):205–251, 1999.

[8] Carr, Peter and Lewis, Keith. Corridor variance swaps. *Risk*, 17(2):67–72, 2004.

[9] Carr, Peter and Madan, Dilip. Towards a theory of volatility trading. In R. Jarrow, editor, *Volatility*, pages 417–427. Risk Publications, 1998.

[10] Carr, Peter and Wu, Liuren. Static hedging of standard options. *Journal of Financial Econometrics*, 12(1):3–46, 2014.

[11] Christoffersen, Peter., Heston, Steven., and Jacobs, Kris. Capturing option anomalies with a variance-dependent pricing kernel. *Review of Financial Studies*, 26(8):1963–2006, 2013.

[12] Cuesdeanu, Horatio and Jackwerth, Jens Carsten. The pricing kernel puzzle: Survey and outlook. *Annals of Finance*, 14(3):289–329, 2018.

[13] Demeterfi, Kresimir., Derman, Emanuel., Kamal, Michael., and Zou, Joseph. A guide to volatility and variance swaps. *Journal of Derivatives*, 6(4):9–32, 1999.

[14] Duffie, Darrell., Pan, Jun., and Singleton, Kenneth. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376, 2000.

[15] Epstein, Larry G and Zin, Stanley E. Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57(4):937–969, 1989.

[16] Fengler, Matthias R and Hin, Lin-Yee. Semi-nonparametric estimation of the call-option price surface under strike and time-to-expiry no-arbitrage constraints. *Journal of Econometrics*, 184(2):242–261, 2015.

[17] Heston, Steven L. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[18] Jackwerth, Jens Carsten. Recovering risk aversion from option prices and realized returns. *Review of Financial Studies*, 13(2):433–451, 2000.

[19] Kahneman, Daniel and Tversky, Amos. Prospect theory: An analysis of decisions under risk. *Econometrica*, 47(2):278–291, 1979.

[20] Klibanoff, Peter., Marinacci, Massimo., and Mukerji, Sujoy. A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849–1892, 2005.

[21] Leland, Hayne E. Who should buy portfolio insurance? *Journal of Finance*, 35(2):581–594, 1980.

[22] Lucas Jr, Robert E. Asset prices in an exchange economy. *Econometrica*, 46(6):1429–1445, 1978.

[23] Martin, Ian. What is the expected return on the market? *Quarterly Journal of Economics*, 132(1):367–433, 2017.

[24] Ohsaki, Shuichi and Yamazaki, Akira. Static hedging of defaultable contingent claims: A simple hedging scheme across equity and credit markets. *International Journal of Theoretical and Applied Finance*, 14(2):239–264, 2011.

[25] Polkovnichenko, Valery and Zhao, Feng. Probability weighting functions implied in options prices. *Journal of Financial Economics*, 107(3):580–609, 2013.

[26] Prelec, Drazen. The probability weighting function. *Econometrica*, 66(3):497–527, 1998.

[27] Rosenberg, Joshua V and Engle, Robert F. Empirical pricing kernels. *Journal of Financial Economics*, 64(3):341–372, 2002.

[28] Schoutens, Wim. Moment swaps. *Quantitative Finance*, 5(6):525–530, 2005.

[29] Shefrin, Hersh. *A Behavioral Approach to Asset Pricing*. Elsevier, 2008.

[30] Song, Zhaogang and Xiu, Dacheng. A tale of two option markets: Pricing kernels and volatility risk. *Journal of Econometrics*, 190(1):176–196, 2016.

[31] Takahashi, Akihiko and Yamazaki, Akira. Efficient static replication of European options under exponential Lévy models. *Journal of Futures Markets*, 29(1):1–15, 2009a.

[32] Takahashi, Akihiko and Yamazaki, Akira. A new scheme for static hedging of European derivatives under stochastic volatility models. *Journal of Futures Markets*, 29(5):397–413, 2009b.

[33] Takahashi, Akihiko., Tsuzuki, Yukihiro., and Yamazaki, Akira. Hedging European derivatives with the polynomial variance swap under uncertain volatility environments. *International Journal of Theoretical and Applied Finance*, 14(04):485–505, 2011.

[34] Tversky, Amos and Kahneman, Daniel. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4):297–323, 1992.

[35] Umezawa, Yuji and Yamazaki, Akira. Pricing path-dependent options with discrete monitoring under time-changed lévy processes. *Applied Mathematical Finance*, 22(2):133–161, 2015.

[36] Wu, George and Gonzalez, Richard. Curvature of the probability weighting function. *Management Science*, 42(12):1676–1690, 1996.

[37] Yamazaki, Akira. A dynamic equilibrium model for U-shaped pricing kernels. *Quantitative Finance*, 18(5):851–875, 2018.

[38] Yamazaki, Akira. Probability weighting and default risk: A possible explanation for distressed stock puzzles. 2019.

本ワーキングペーパーの掲載内容については、著編者が責任を負うものとします。

法政大学イノベーション・マネジメント研究センター
The Research Institute for Innovation Management, HOSEI UNIVERSITY

〒102-8160 東京都千代田区富士見 2-17-1
TEL: 03(3264)9420　FAX: 03(3264)4690
URL: http://riim.ws.hosei.ac.jp
E-mail: cbir@adm.hosei.ac.jp