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Abstract

This paper proposes a general method to recover the subjective probability distribution of nonlinear payoffs from option prices. We show that the characteristic function of the distribution can be represented as the present value of a static option portfolio with complex-valued portfolio weights. By applying Fourier inversion, we derive the subjective probability distribution from the characteristic function. As an illustration, we successfully recover the subjective probability distributions of option payoffs and agent's utility. This research contributes a valuable framework for understanding subjective probability distributions and their implications for financial analysis and decision-making.

Keywords: subjective probability distribution, option payoff, utility function, pricing kernel, static replication
Classification codes: G12, G13

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1 Introduction

The standard theory of asset pricing posits that asset prices are influenced by the subjective probabilities held by a representative agent. However, in financial markets, these subjective probabilities are not directly observable. This paper introduces a novel approach to recovering the subjective probability distributions associated with nonlinear payoffs, utilizing observed option prices in markets. Our proposed method offers a general framework for achieving this goal. To demonstrate the effectiveness of our method, we provide illustrative examples that demonstrate its application in recovering the subjective probability distributions of option payoffs, as well as those of the utility preferences of the agent.

Recovering subjective probability distributions from asset price data is crucial for gaining insights into market sentiment and understanding ex-ante probability distributions. A notable contribution in this area is Ross's (2015) introduction of the Ross recovery theorem, which offers a method to extract the subjective probabilities of an asset price from its option prices. While the Ross recovery theorem is groundbreaking, it faces several implementation challenges. First, Ross's method relies on knowing all risk-neutral *transition* probabilities, which can be difficult to obtain. Jackwerth and Menner (2018) have highlighted the complexity of extracting a risk-neutral probability transition matrix from option prices, resulting in unstable estimates of subjective probability distributions. Second, Ross's method recovers discrete probability distributions instead of continuous ones, which may limit practical applicability in many cases. Third, Ross's method assumes that the pricing kernel is the ratio of the marginal utility in the future state to the marginal utility in the present state, potentially imposing restrictive assumptions.

Another estimation method for recovering the subjective probability distribution of asset prices has been proposed by Yamazaki (2022), which is based on a static replication strategy using option portfolios. Unlike Ross's method, Yamazaki's approach yields continuous probability distributions and does not impose any specific assumptions on the pricing kernel. Furthermore, Yamazaki empirically examined the behavior of the subjective probability distributions of the S&P 500 index returns. Although Yamazaki's method is novel in itself, the results obtained through his approach can be derived from previous methods such as Bliss and Panigirtzoglou (2004), who utilized the well-known three-way relationship between the subjective density function, the risk-neutral density function, and the pricing kernel to estimate the coefficient of risk aversion.

This paper focuses on developing a recovery method specifically for the subjective probability distribution of a nonlinear payoff on an asset, as opposed to the subjective probability distribution of the asset price itself. It is important to note that Ross (2015), Yamazaki (2022), and Bliss and Panigirtzoglou (2004) address the latter case. As far as our knowledge extends, this is the first endeavor to recover the subjective probability distribution of a nonlinear payoff.

A rough sketch for developing our method is as follows. We deal with the characteristic function of the subjective probability distribution of a nonlinear payoff. This characteristic function can be thought of as the subjective expectation of a complex-valued payoff. By using the reciprocal kernel proposed by Yamazaki (2022), the subjective expectation can be transformed into the risk-neutral expectation of the complex-valued payoff multiplied by the reciprocal kernel. Then, with the static replication strategy, the risk-neutral expectation can be expressed as the present value of a static portfolio consisting of plain vanilla options with complex-valued portfolio weights. As a result, the characteristic function we wish to obtain can be obtained from the option prices observed in the option market. The inversion formula for the characteristic function allows us to recover the subjective probability distribution of the nonlinear payoff.

There are two key points in our method: One of them is the reciprocal kernel. Yamazaki (2022) considered the projection of a pricing kernel onto a future asset price and defined the reciprocal kernel as the reciprocal of the projected pricing kernel. He proved that the subjective expectation of an arbitrary payout depending on the terminal asset price equals the present value of the product of the reciprocal kernel and the payout. He then applied the reciprocal kernel to derive the formula for the subjective cumulative distribution function of an asset price. A similar approach was used by Chabi-Yoa and Loudis (2020). Similarly, we utilize the reciprocal kernel to transform subjective expectations into risk-neutral expectations.

Another key point is the static replication strategy. Basically, there are two ways to use the static replication strategy. First, it is applied to replicate contingent claims. For example, Carr et al. (1998), Fink (2003), Nalholm and Poulsen (2006) developed static replication strategies for barrier options, while Takahashi and Yamazaki (2009a, 2009b) and Carr and Wu (2014) did them for long-term options. Osaki and Yamazaki (2011) proposed a method to replicate defaultable bonds by a static portfolio of plain vanilla options. Many researchers, including Carr and Madan (1998), Demeterfi et al. (1999), Carr and Lewis (2004), and Takahashi et al. (2011), also studied semi-static replication methods for volatility derivatives such as variance swaps and gamma swaps. Second, static replication strategies have been used in methods to estimate fundamental statistics of asset prices from observed option prices. For example, Bakshi et al. (2003), Martin (2017), and Yamazaki (2022) developed such estimation methods. This research belongs to the second case and is novel in that it considers static option portfolios with *complex-valued* portfolio weights.

This paper focuses on the recovery of subjective probability distributions of two types of payoffs: option payoffs and agent's utility.

It is widely recognized that any arbitrary payoff can be expressed as a linear combination of call and put option payoffs. The call and put option payoffs serve as fundamental components for constructing nonlinear payoffs. In a model-free framework, based on reasonable assumptions, we prove that the characteristic functions of the subjective probability distributions of call and put payoffs can be represented as the present values of static option portfolios with complex-valued portfolio weights. Consequently, the subjective probability distributions can be derived from these characteristic functions by employing inversion formulas such as Lévy's inversion theorem and Gil-Pelaez's theorem. To validate the efficacy of our recovery method, we conduct a numerical test employing the Black-Scholes model. In this test, we set the market price of risk, establish both a physical and a risk-neutral world, and attempt to recover the physical probability distribution of an at-the-money straddle payoff using our proposed method. Furthermore, we verify the consistency of the recovered physical distribution with the distribution obtained from a Monte Carlo simulation conducted with the Black-Scholes model.

In our empirical experiment, the objective is to recover the subjective probability distributions of agent's utility with constant relative risk aversion (CRRA), which includes the risk-neutral case. To achieve this, we consider a textbook-style optimal investment-consumption problem and suppose CRRA-utility functions in relation to the market portfolio's levels. To estimate the shape of the subjective probability distributions associated with these utility functions, we employ implied volatility data of the S&P 500 index, which we regard as a suitable proxy for the market portfolio.

The subsequent sections of this paper are organized as follows: Section 2 provides a comprehensive description of the model-free setup, taking into account several mild assumptions. Section 3 outlines the recovery formulas developed for obtaining the subjective probability distributions of option payoffs. Section 4 presents the empirical experiment conducted to recover the subjective probability distributions of agent's utility. Section 5 concludes the research by summarizing the key findings and implications. Furthermore, Appendices contain all the proofs and some supplements.

2 Setup

This section introduces a model-free framework for recovering the subjective probability distributions associated with nonlinear payoffs of an asset from its option prices. We assume that no arbitrage opportunities exist and that all plain vanilla options on the asset are tradable without incurring transaction costs. Consider a fixed period, $[0, T]$, and let S_t be the asset price at time $t \in [0, T]$.

2.1 Option payoffs

Call and put options play a crucial role in our recovery method, serving as fundamental components. Their payoffs form the foundational building blocks for all nonlinear payoffs. Moreover, their prices serve as a valuable source for recovering the subjective probability distribution of an arbitrary nonlinear payoff. Call and put option payoffs are

$$(S_T - K)_+ \qquad \text{and} \qquad (K - S_T)_+, \tag{2.1}$$

respectively, where K is a strike price and T is option maturity.

The prices of the call and put options as a function of the strike price K are expressed as

$$C(K) := \frac{1}{R_f}\mathbb{E}_*\left[(S_T - K)_+\right] \qquad \text{and} \qquad P(K) := \frac{1}{R_f}\mathbb{E}_*\left[(K - S_T)_+\right], \tag{2.2}$$

respectively, where $\mathbb{E}_*[\,\cdot\,]$ denotes the expectation operator under the risk-neutral probability measure, $\mathbb{Q}$, and R_f is the gross return on a risk-free asset during the period. Let F be the forward price of the asset expiring at time T. This paper deals primarily with the out-of-the-money (OTM) or forward at-the-money (ATM) options.

2.2 Reciprocal kernel

Let $\mathcal{M}$ be a strictly positive random variable, such that the price of an arbitrary payoff X paid at time T is given by

$$\mathbb{E}\left[\mathcal{M}X\right]. \tag{2.3}$$

where $\mathbb{E}[\,\cdot\,]$ denotes the expectation operator under the subjective probability measure, denoted as $\mathbb{P}$. Here, the subjective probability represents the belief or consensus of the representative agent or market participants. Namely, $\mathcal{M}$ is a pricing kernel[1]. The arbitrage-free condition ensures the existence of a strictly positive pricing kernel.

The projection of the pricing kernel onto S_T is defined as

$$\mathcal{M}(S_T) := \mathbb{E}\left[\mathcal{M} \,|\, S_T\right]. \tag{2.4}$$

[1]Employing the pricing kernel, the option prices can also be written as

$$C(K) = \mathbb{E}\left[\mathcal{M}(S_T - K)_+\right] \qquad \text{and} \qquad P(K) = \mathbb{E}\left[\mathcal{M}(K - S_T)_+\right].$$

Additionally, it is critical to emphasize that the expectation operators in the above option pricing equations are under the subjective probability measure $\mathbb{P}$.

We assume that the projected pricing kernel, $\mathcal{M}(x)$, is twice differentiable for any $x > 0$.

The shape of $\mathcal{M}(x)$ is largely controversial. The classical asset pricing theory states that the pricing kernel projected onto the market portfolio decreases monotonically. However, several researchers have reported that such a theoretical result is inconsistent with empirical analyses. For example, via an empirical estimation based on the S&P 500 index, Jackwerth (2000, 2004) and Yamazaki (2022) obtained tilde-shaped pricing kernels. However, Bakshi et al. (2010) and Christoffersen et al. (2013) proved that empirical pricing kernels are U-shaped. Yamazaki (2020) demonstrated that the U-shaped price kernel offers the possibility of explaining the distressed stock puzzles that are anomalous patterns in financially distressed stock prices. Concurrently, Chaudhuri and Schroder (2015) reported that empirical pricing kernels projected onto individual stocks exhibited downward slopes in the whole range. The scope of this paper does not cover the appropriate shape of a pricing kernel because it may depend on the choice of an asset onto which the pricing kernel is projected.

Next, the reciprocal of $\mathcal{M}(x)$, which Yamazaki (2022) calls the *reciprocal kernel*, is defined as

$$\mathcal{N}(x) := 1/\mathcal{M}(x). \tag{2.5}$$

The reciprocal kernel is equivalent to $\mathcal{M}(x)$ in effect. However, it is convenient to apply the reciprocal kernel, $\mathcal{N}(x)$, rather $\mathcal{M}(x)$. Note that the reciprocal kernel is also strictly positive and twice differentiable under our assumptions.

3 Subjective probability distributions

This section describes a method to recover the subjective probability distributions of nonlinear payoffs. While the recovery method discussed below is applicable in a general context, we focus on option payoffs as an illustrative example of nonlinear payoffs throughout this section.

3.1 Characteristic functions

We apply the characteristic function approach to recover a subjective probability distribution. The characteristic function of the distribution is represented as *subjective* expected value. To begin, we transform the subjective expectation into the risk-neutral expectation using the following lemma, which plays a vital role in our recovery method. For the proof of this lemma, see Appendix A.1 of Yamazaki (2022).

Lemma 1 (Yamazaki, 2022) *Let $H(S_T)$ be an arbitrary payout depending on the time-T price of the underlying asset. Then, the subjective expected value of $H(S_T)$ equals the price of the product of the reciprocal kernel $\mathcal{N}(S_T)$ and $H(S_T)$. That is,*

$$\mathbb{E}\left[H(S_T)\right] = \frac{1}{R_f}\mathbb{E}_*\left[\mathcal{N}(S_T)H(S_T)\right].$$

Next, we utilize a static replication strategy to express the transformed characteristic function as the present value of a static option portfolio. This portfolio is comprised of plain vanilla options with portfolio weights that are complex-valued. The following propositions offer the characteristic functions of the subjective probability distributions of call and put option payoffs. For the detailed proofs of these propositions, refer to Appendix A.

Proposition 1 (Characteristic function of call payoff) *Let*

$$\Phi_C(\theta) := \mathbb{E}\left[e^{i\theta(S_T-K)_+}\right], \tag{3.1}$$

be the characteristic function of the subjective probability distribution of a call option payoff, where $i := \sqrt{-1}$ is the imaginary unit. For $K > F$, it is represented as follows:

$$\begin{aligned}\Phi_C(\theta) = & A_C(K) + i\theta\mathcal{N}(K)C(K) \\ & + \int_K^\infty \left[\mathcal{N}''(x) + 2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right] e^{i\theta(x-K)}C(x)dx,\end{aligned} \tag{3.2}$$

where $A_C(K)$ is a function of the strike price K, independent of the parameter θ, expressed as

$$A_C(K) := \frac{1}{R_f}\mathcal{N}(F) + \int_0^F \mathcal{N}''(x)P(x)dx + \int_F^K \mathcal{N}''(x)C(x)dx. \tag{3.3}$$

The representation (3.2) demonstrates that the characteristic function of the subjective probability distribution can be expressed as the present value of a static option portfolio with complex-valued portfolio weights. This portfolio consists of the the following components: (i) a long position in $\mathcal{N}(F)$ units of the risk-free asset; (ii) a long position in $\mathcal{N}''(x)dx$ units of put options with all strikes x smaller than F; (iii) a long position in $\mathcal{N}''(x)dx$ units of call options with all strikes x within the range of (F, K); (iv) a long position in $i\theta\mathcal{N}(K)$ units of a call option with strike K; (v) a long position in $\left[\mathcal{N}''(x) + 2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right] e^{i\theta(x-K)}dx$ units of call options with all strikes x larger than K.

Proposition 2 (Characteristic function of put payoff) *Let*

$$\Phi_P(\theta) := \mathbb{E}\left[e^{i\theta(K-S_T)_+}\right], \tag{3.4}$$

be the characteristic function of the subjective probability distribution of a put option payoff. For $K < F$, it is represented as follows:

$$\begin{aligned}\Phi_P(\theta) &= A_P(K) + i\theta\mathcal{N}(K)P(K) \\ &\quad + \int_0^K \left[\mathcal{N}''(x) - 2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right] e^{i\theta(K-x)}P(x)dx,\end{aligned} \tag{3.5}$$

where $A_P(K)$ is a function of the strike price K, independent of the parameter θ, expressed as

$$A_P(K) := \frac{1}{R_f}\mathcal{N}(F) + \int_K^F \mathcal{N}''(x)P(x)dx + \int_F^\infty \mathcal{N}''(x)C(x)dx. \tag{3.6}$$

The representation (3.5) has interpretations similar to those of (3.2).

In the final step, we employ the inverse Fourier transform to derive the subjective probability density function from the corresponding characteristic function. Alternatively, we can also apply Gil-Pelaez's theorem (Wendel, 1961), an inversion formula, to obtain the cumulative distribution function from the characteristic function.

3.2 Black-Scholes test

In this subsection, we aim to demonstrate the effectiveness of our recovery method. To confirm the validity of our approach, we implement a purely numerical simulation within the framework of the Black-Scholes model (Black and Scholes, 1973). This simulation serves as a preliminary step for the empirical experiment that will be discussed in Section 4.

Suppose that the projected pricing kernel has the form

$$\mathcal{M}(S_T) = e^{-\delta T}\left(\frac{S_T}{S_0}\right)^{-\gamma}, \tag{3.7}$$

where δ is the rate of time preference and γ represents the modified coefficient of relative risk aversion. Under the subjective probability measure $\mathbb{P}$, the Black-Scholes model is described as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{3.8}$$

where μ is the expected rate of return on the asset, σ represents the volatility of the asset, and W is the standard Brownian motion. In equilibrium, the expected rate of return can be expressed as

$$\mu = \frac{\delta}{1-\gamma} + \frac{1}{2}\sigma^2\gamma.$$

The risk-free interest rate, which serves as the drift term of the Black-Scholes model under the risk-neutral probability measure $\mathbb{Q}$, can be expressed as

$$r = \frac{\delta}{1-\gamma} - \frac{1}{2}\sigma^2\gamma.$$

For more information on the above settings, see Yamazaki (2018).

Our purpose is to recover the subjective probability distribution of a straddle payoff within the framework of the Black-Scholes model (3.8) with the projected pricing kernel (3.7). The straddle payoff is defined as

$$(S_T - F)_+ + (F - S_T)_+,$$

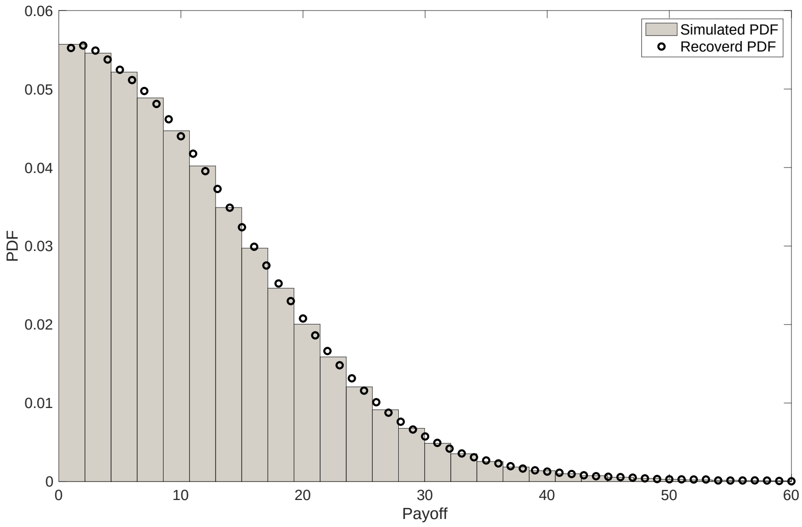
where the strike price is set to the forward price F. The characteristic function of the subjective probability distribution of this payoff is provided in Appendix B.1. The parameter values used for the test are listed in Table 1. In this setting, the expected rate of return on the asset, μ, is 3.46%, while the risk-free interest rate, r, is 2.26%. In all the implementation of this paper, we use MATLAB R2022b as numerical computation software.

Figure 1 presents the results of the Black-Scholes test, showing the subjective probability distributions. In Panel A, we exhibit the simulated probability density function and the recovered probability density function. The simulated density function is obtained by conducting a Monte Carlo simulation with 1 million samples, while the recovered density function is calculated using our recovery method. To obtain the recovered probability density function, we employ the inverse Fourier transform of the characteristic function (B.1). The numerical inversion is performed using the Gauss-Kronrod quadrature method implemented in the MATLAB function *quadgk*.

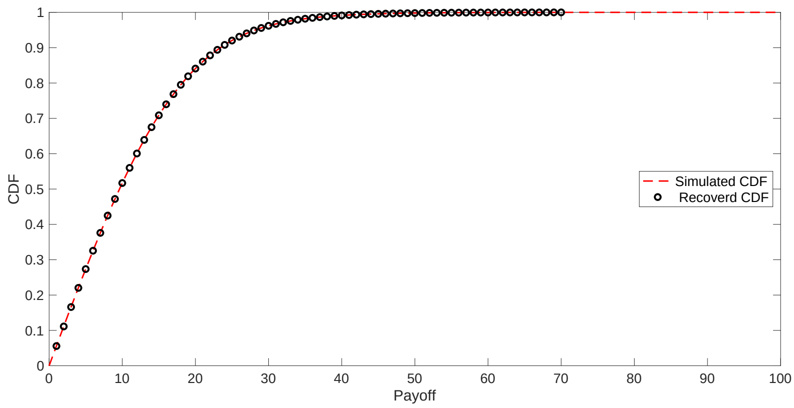
In Panel B, we show the simulated cumulative distribution function and the recovered cumulative distribution function that are obtained by procedures similar to those in Panel A. Gil-Pelaez's theorem in conjunction with the Gauss-Kronrod quadrature is applied to obtain the recovered cumulative distribution function.

Through the Black-Scholes test, we have effectively demonstrated the efficacy of our recovery method.

Figure 1. Subjective probability distributions of straddle payoff



Panel A: Probability density functions



Panel B: Cumulative distribution functions

Panel A exhibits subjective probability density functions of a straddle payoff under the Black-Scholes model. The simulated probability density function is obtained through a Monte Carlo simulation with 1 million samples, while the recovered probability density function calculated using our recovery method. In Panel B, we plot simulated and recovered cumulative distribution functions under the same setting as in Panel A.

Table 1. Parameters for Black-Scholes test

S_0	δ	γ	σ	T
100	0.02	0.3	0.2	0.5

4 Empirical experiment

This section presents an empirical experiment in which the subjective probability distribution of agent's utility is obtained from option prices observed in the real market, using our recovery method.

4.1 Utility function

We consider a simple single-period consumption-investment problem for a representative agent with a utility function, $U(x)$. In this context, the pricing kernel is given by

$$\mathcal{M} = \xi \frac{U'(C_T^{op})}{U'(C_0^{op})}, \tag{4.1}$$

where $\xi \in (0,1)$ is the time preference discount factor, and C_t^{op} is the optimal consumption for the representative agent at time $t \in \{0, T\}$. Suppose that the representative agent has the CRRA-utility function in terms of consumption defined by

$$U(x) = \begin{cases} a_1 \dfrac{x^{1-\eta}}{1-\eta} + a_2 & \eta \geq 0,\ \eta \neq 1 \\ \\ \log x + a_3 & \eta = 1, \end{cases} \tag{4.2}$$

where η is the coefficient of relative risk aversion, and a_1, a_2, and a_3 are some constants for positive affine transforms. In equilibrium, the terminal optimal consumption is equal to the terminal total value of the market portfolio, represented as $C_T^{op} = S_T(1+d)$. Here, S_t is the value of the market portfolio at time t, and d is some constant representing the dividend yield of the market portfolio. Consequently, the terminal level of the agent's utility can be viewed as a nonlinear payoff on the market portfolio. To account for this, we redefine the CRRA-utility function in terms of the market portfolio as follows.

$$u(S_T) := U(S_T(1+d)) = U(C_T^{op}). \tag{4.3}$$

The objective of this section is to recover the subjective probability distribution for the utility function (4.3). For standardization, we set a_1, a_2, a_3 in (4.2) such that the utility functions with different coefficients of relative risk aversion take zero and the same slope at $S_T = S_0$. The concrete expressions of a_1, a_2, and a_3 can be found in Appendix B.2. Figure 2 depicts CRRA-utility functions ($\eta = 1, 3, 5$), including the risk-neutral case ($\eta = 0$), as used in our empirical experiment.

In this setting, the reciprocal kernel is given by

$$\mathcal{N}(S_T) = G_\eta g_\eta(S_T),$$

where $g_\eta(x) = x^\eta$ and G_η is a constant defined by

$$G_\eta := \xi^{-1}\left(\frac{1+d}{C_0^{op}}\right)^\eta .$$

To determine the value of G_η, it is not essential to have prior knowledge of the time preference discount factor ξ and the initial optimal consumption C_0^{op}. Yamazaki (2022) proposed a method to obtain this value using the following formula.

$$\frac{1}{G_\eta} = \frac{1}{R_f} g_\eta(F) + \int_0^F g_\eta''(x)P(x)dx + \int_F^\infty g_\eta''(x)C(x)dx.$$

The characteristic function of the subjective probability distribution for the utility function (4.3) is defined as

$$\Phi_u(\theta) := \mathbb{E}\left[e^{i\theta u(S_T)}\right]. \tag{4.4}$$

See Appendix B.3 for the static portfolio representation of the characteristic function (4.4). We apply the numerical inversion of the characteristic function (4.4) to obtain the subjective probability density function of the agent's utility, employing the same numerical procedure as described in Section 3.2.

4.2 Data and results

In our empirical experiment, we consider the S&P 500 index as the market portfolio. The data for plain vanilla options on the S&P 500 index were obtained from the Cboe DataShop website[2]. We downloaded the Black-Scholes implied volatilities based on end-of-day options mid quotes from the Chicago Board Options Exchange. The dataset comprises implied volatilities of virtual S&P 500 index options with constant maturities and various strikes defined by moneyness.

For our experiment, we selected implied volatilities with a maturity of 180 days observed on December 19, 2011, and January 30, 2019. The former represents a case of relatively high volatility, while the latter represents a case of relatively low

[2]`https://datashop.cboe.com/option-eod-summary`

Figure 2. CRRA-utility functions

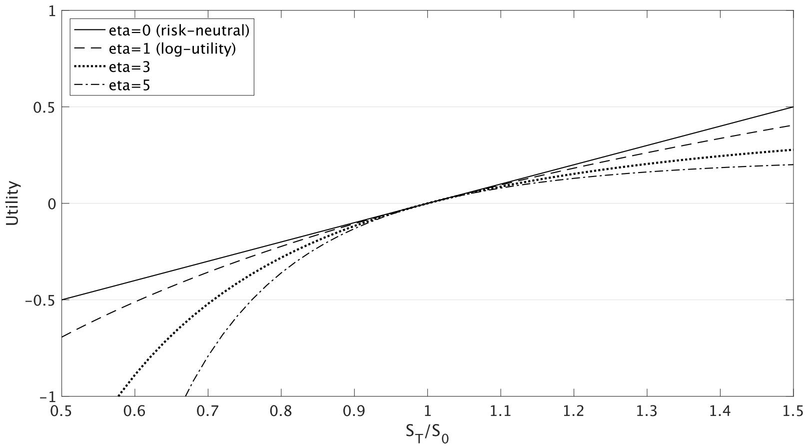


Figure 2 depicts CRRA-utility functions with different coefficients of relative risk aversion, $\eta = 0, 1, 3, 5$. To ensure that these utility functions take zero and the same slope at $S_T/S_0 = 1$, we apply positive affine transforms to them.

volatility. Figure 3 displays the implied volatilities observed on these two specific trading days.

Following the approach introduced by Yamazaki (2022), we utilize a quadratic polynomial approximation to construct an end-of-day implied volatility curve. This curve is obtained by fitting observed implied volatilities with respect to moneyness using the least-squares method. The resulting implied volatility curve is then incorporated into the Black-Scholes option pricing formula to calculate the option prices required for the recovery formula.

It is important to acknowledge that this interpolation method may not be the optimal choice for capturing the intricacies of observed implied volatilities. However, alternative interpolation and extrapolation methods, such as cubic spline interpolation and piecewise Hermite interpolation, have been found to lead to overfitting and fail to recover a smooth single-peak probability density function. Consequently, the quadratic polynomial approximation is employed to strike a balance between simplicity and avoiding overfitting issues[3].

The interest rate data required for the Black-Scholes option pricing formula and the gross return on the risk-free asset are obtained from the U.S. Department

[3]Upon request, we can provide the experiment results in the cases that cubic spline interpolation and piecewise Hermite interpolation are applied to construct implied volatility curves. However, these results have not exhibited the desired level of subjective density functions, as anticipated.

of the Treasury website[4]. To obtain the implied dividend yields of the S&P 500 index, which are used as the dividend yield d, we employ the put-call parity based on near-the-money option prices. This procedure is consistent with the approach employed by Aït-Sahalia and Lo (1998) and Polkovnichenko and Zhao (2013).

Figure 3. Implied volatilities of options on S&P 500 index

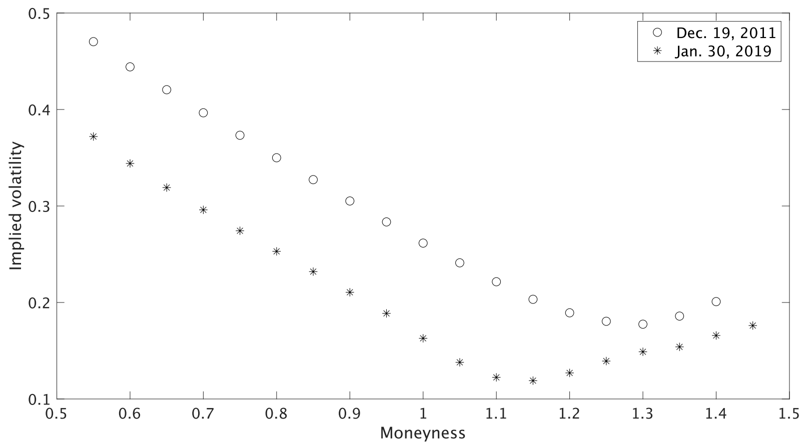


Figure 3 plots the implied volatilities of options on the S&P 500 index, as observed on December 19, 2011, and January 30, 2019. The dataset is sourced from Cboe DataShop.

Figure 4 exhibits the subjective probability density functions of agent's utility in terms of the level of the S&P 500 index. These density functions are derived using our recovery method. Panel A displays the subjective probability density functions with $\eta = 0, 1, 3, 5$, obtained from the S&P 500 implied volatilities observed on December 19, 2011, while Panel B exhibits the density functions based on data from January 30, 2019.

It is important to note that the representative agent exhibits not only different forms of utility functions but also distinct subjective probability distributions of the future level of the S&P 500 index level, depending on the coefficient of relative risk aversion η. For example, as the representative agent becomes more risk-averse, the subjective expected return on the S&P 500 index increases. For more details on the characteristics of the subjective probability distributions of the level of the S&P 500 index, refer to Yamazaki (2022). Additionally, as the representative agent becomes more risk-averse, the marginal utility diminishes at a faster rate. Consequently, the recovered subjective probability distributions of agent's utility are depicted in Figure 4.

[4]`https://home.treasury.gov`

The presence of wavy right tails in the probability density functions observed in Panel B can be attributed to instability in the numerical Fourier inversion. We acknowledge that addressing this issue remains an open problem in our research.

5 Conclusion

In this research, we develop a general method for recovering the subjective probability distribution of the nonlinear payoff of a financial asset. We demonstrate that the characteristic function of the subjective probability distribution can be expressed as the present value of a static option portfolio with complex-valued portfolio weights. The subjective probability distribution can be obtained using a numerical inversion technique. To validate our recovery method, we conduct a Black-Scholes test that confirms its ability to accurately estimate the subjective probability distribution of a straddle payoff. Furthermore, we conduct an empirical experiment using implied volatilities of the S&P 500 index to estimate the subjective probability distribution of agent's utility.

This research contributes a valuable framework for understanding subjective probability distributions and their implications for financial analysis and decision-making. However, we acknowledge that our recovery method is currently limited in its application, specifically in cases where the nonlinear payoffs are contingent on the terminal price of an asset. Future research directions include extending our method to recover the subjective probability distribution of path-dependent payoffs, such as variance swaps, which involve payoffs dependent on the realized variance of an underlying asset, as well as time-inseparable utility functions, which capture preferences that are not solely determined by the terminal wealth. These are examples of payoffs that depend on the path taken by the underlying asset, and expanding our method to encompass these scenarios will broaden its applicability and significance.

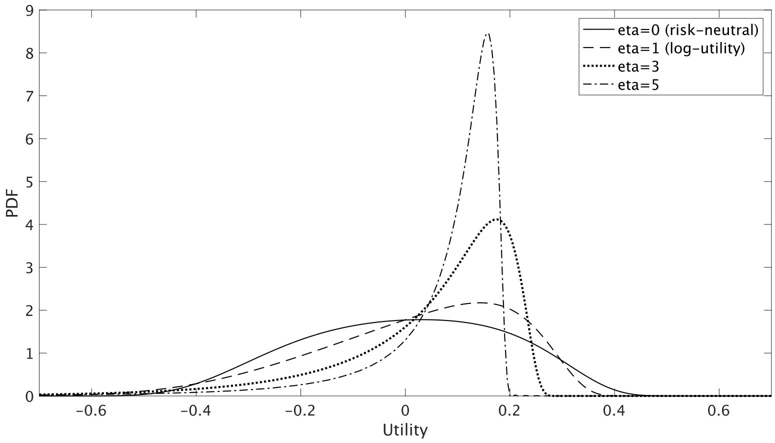
A Proofs

A.1 Proof of Proposition 1

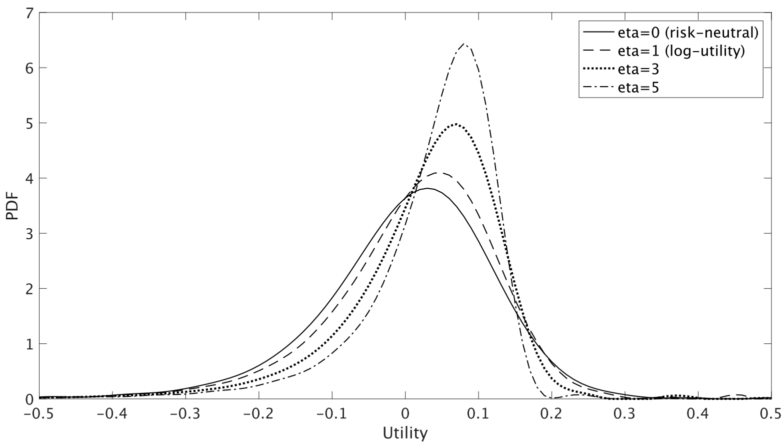
Let $G(S_T) := e^{i\theta(S_T-K)_+}$ and $g(S_T) := \mathcal{N}(S_T)G(S_T)$. According to Carr and Madan (1998), the following holds:

$$g(S_T) = g(F) + g'(F)(S_T - F) + \int_0^F g''(x)(x - S_T)_+ dx + \int_F^\infty g''(x)(S_T - x)_+ dx.$$

Figure 4. Subjective probability distributions of agent's utility



Panel A: December 19, 2011



Panel B: January 30, 2019

Figure 4 exhibits the subjective probability density functions for CRRA-utility functions as they relate to the level of the S&P 500 index. These density functions are obtained using our recovery method, which utilizes the numerical Fourier inversion. Panel A depicts the density functions derived from the S&P 500 implied volatilities observed on December 19, 2011, while Panel B shows those based on the data from January 30, 2019.

Employing Lemma 1 and $K > F$, the subjective expected value of $G(S_T)$ can be expressed as follows:

$$\begin{aligned}
\mathbb{E}\left[G(S_T)\right] &= \frac{1}{R_f}\mathbb{E}_*\left[g(S_T)\right] \\
&= \frac{1}{R_f}g(F) + \int_0^F g''(x)P(x)dx + \int_F^\infty g''(x)C(x)dx. \qquad \text{(A.1)}
\end{aligned}$$

Note that

$$g''(x) = \left\{\mathcal{N}''(x) + \left[2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right]\mathbf{1}_{\{x>K\}} + i\theta\mathcal{N}(x)\delta(x-K)\right\}G(x),$$

where $\delta(x)$ is the Dirac delta function. Therefore, the following is obtained:

$$\begin{aligned}
\int_0^F g''(x)P(x)dx &= \int_0^F \mathcal{N}''(x)G(x)P(x)dx \\
&\quad + \int_0^F \left[2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right]\mathbf{1}_{\{x>K\}}G(x)P(x)dx \\
&\quad + \int_0^F i\theta\mathcal{N}(x)\delta(x-K)G(x)P(x)dx \\
&= \int_0^F \mathcal{N}''(x)P(x)dx, \qquad \text{(A.2)}
\end{aligned}$$

and

$$\begin{aligned}
\int_F^\infty g''(x)C(x)dx &= \int_F^\infty \mathcal{N}''(x)G(x)C(x)dx \\
&\quad + \int_F^\infty \left[2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right]\mathbf{1}_{\{x>K\}}G(x)C(x)dx \\
&\quad + \int_F^\infty i\theta\mathcal{N}(x)\delta(x-K)G(x)C(x)dx \\
&= \int_F^K \mathcal{N}''(x)C(x)dx + \int_K^\infty \mathcal{N}''(x)e^{i\theta(x-K)}C(x)dx \\
&\quad + \int_K^\infty \left[2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right]e^{i\theta(x-K)}C(x)dx \\
&\quad + i\theta\mathcal{N}(x)C(x). \qquad \text{(A.3)}
\end{aligned}$$

Substituting (A.2) and (A.3) into (A.1) yields (3.2). □

A.2 Proof of Proposition 2

Let $H(S_T) := e^{i\theta(K-S_T)_+}$ and $h(S_T) := \mathcal{N}(S_T)H(S_T)$. Similar to the proof of Proposition 2, the subjective expected value of $H(S_T)$ for $K < F$ can be expressed as follows:

$$\begin{aligned}\mathbb{E}\left[H(S_T)\right] &= \frac{1}{R_f}\mathbb{E}_*\left[h(S_T)\right] \\ &= \frac{1}{R_f}h(F) + \int_0^F h''(x)P(x)dx + \int_F^\infty h''(x)C(x)dx. \qquad (A.4)\end{aligned}$$

Note that

$$h''(x) = \left\{\mathcal{N}''(x) - \left[2i\theta\mathcal{N}'(x) + \theta^2\mathcal{N}(x)\right]\mathbf{1}_{\{x\le K\}} + i\theta\mathcal{N}(x)\delta(x-K)\right\}H(x).$$

Therefore, the following is obtained:

$$\begin{aligned}\int_0^F h''(x)P(x)dx &= \int_0^F \mathcal{N}''(x)H(x)P(x)dx \\ &\quad - \int_0^F \left[2i\theta\mathcal{N}'(x) + \theta^2\mathcal{N}(x)\right]\mathbf{1}_{\{x\le K\}}H(x)P(x)dx \\ &\quad + \int_0^F i\theta\mathcal{N}(x)\delta(x-K)H(x)P(x)dx \\ &= \int_0^K \mathcal{N}''(x)e^{i\theta(K-x)}P(x)dx + \int_K^F \mathcal{N}''(x)P(x)dx \\ &\quad - \int_0^K \left[2i\theta\mathcal{N}'(x) + \theta^2\mathcal{N}(x)\right]e^{i\theta(K-x)}P(x)dx \\ &\quad + i\theta\mathcal{N}(x)P(x), \qquad (A.5)\end{aligned}$$

and

$$\begin{aligned}\int_F^\infty h''(x)C(x)dx &= \int_F^\infty \mathcal{N}''(x)H(x)C(x)dx \\ &\quad - \int_F^\infty \left[2i\theta\mathcal{N}'(x) + \theta^2\mathcal{N}(x)\right]\mathbf{1}_{\{x\le K\}}H(x)C(x)dx \\ &\quad + \int_F^\infty i\theta\mathcal{N}(x)\delta(x-K)H(x)C(x)dx \\ &= \int_F^\infty \mathcal{N}''(x)C(x)dx. \qquad (A.6)\end{aligned}$$

Substituting (A.5) and (A.6) into (A.4) yields (3.5). □

B Supplements

B.1 Characteristic function of straddle payoff

Let

$$\Phi_{\mathrm{Str}}(\theta) := \mathbb{E}\left[e^{i\theta[(S_T-F)_+ + (F-S_T)_+]}\right],$$

be the characteristic function of the subjective probability distribution of a straddle payoff with strike price F. It can be expressed as follows:

$$\begin{aligned}\Phi_{\mathrm{Str}}(\theta) = & \frac{1}{R_f}\mathcal{N}(F) + 2i\theta\mathcal{N}(F)C(F) \\ & + \int_0^F \left[\mathcal{N}''(x) - 2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right] e^{i\theta(F-x)}P(x)dx \\ & + \int_F^\infty \left[\mathcal{N}''(x) + 2i\theta\mathcal{N}'(x) - \theta^2\mathcal{N}(x)\right] e^{i\theta(x-F)}C(x)dx. \end{aligned} \quad \text{(B.1)}$$

The representation (B.1) can be derived by a similar procedure to that employed in Propositions 1 and 2.

B.2 Positive affine transform of CRRA-utility functions

We set

$$a_1 = [S_0(1+d)]^{\eta-1}, \qquad a_2 = -\frac{1}{1-\eta}, \qquad a_3 = -\log S_0(1+d),$$

to ensure that CRRA-utility functions with different coefficients of relative risk aversion take zero and the same slope at $S_T = S_0$. Note that the slope is $1/S_0$.

B.3 Characteristic function of agent's utility

The characteristic function of the subjective probability distribution for CRRA-utility function (4.4) can be expressed as follows:

$$\Phi_u(\theta) = \frac{1}{R_f}\mathcal{N}(F)e^{i\theta u(F)} + \int_0^F \Psi(x,\theta)P(x)dx + \int_F^\infty \Psi(x,\theta)C(x)dx, \quad \text{(B.2)}$$

where

$$\Psi(x,\theta) := \left\{\mathcal{N}''(x) + 2i\theta u'(x)\mathcal{N}'(x) + \left[i\theta u''(x) - \theta^2 u'(x)^2\right]\mathcal{N}(x)\right\} e^{i\theta u(x)}.$$

Note that

$$\mathcal{N}'(x) = \eta G_\eta x^{\eta-1}, \qquad \mathcal{N}''(x) = \eta(\eta-1)G_\eta x^{\eta-2},$$

and

$$u'(x) = S_0^{\eta-1} x^{-\eta}, \qquad u''(x) = -\eta S_0^{\eta-1} x^{-\eta-1}.$$

The representation (B.2) can be derived by a similar procedure to that employed in Propositions 1 and 2.

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